Elliptic Genera and Applications to Mirror Symmetry

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Abstract

The paper contains a proof that elliptic genus of a Calabi-Yau manifold is a Jacobi form, finds in which dimensions the elliptic genus is determined by the Hodge numbers and shows that elliptic genera of a Calabi-Yau hypersurface in a toric variety and its mirror coincide up to sign. The proof of the mirror property is based on the extension of elliptic genus to Calabi-Yau hypersurfaces in toric varieties with Gorenstein singularities.

1 Introduction

One of the motivations for this paper was an attempt to understand those invariants of Calabi-Yau manifolds which value on the mirror X^* is determined by the value on original manifold X. Examples of such invariants that attracted the most attention are topological Euler characteristic, Hodge numbers and various d-point functions. In particular the relation between the different kinds of d-point functions yield the famous predictions for enumerative geometry (cf. [6]).

The invariant considered in this paper is elliptic genus for Calabi-Yau manifolds. Elliptic genus of oriented differentiable manifolds first appeared in the works of Landweber-Stong and Ochanine (cf. [18] and references there) as part of attempts to find genera satisfying certain topological conditions and also as a mean for constructing new generalized cohomology theories (elliptic cohomology). Elliptic genus defined in such way is a certain homomorphism from the ring of oriented cobordisms Ω^*_{SO} into the ring of modular forms for the group $\Gamma_0(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \,\mathrm{mod} 2\}$. The value of elliptic genus at the cusps of $\Gamma_0(2)$ is equal to $\hat{A}$-genus and the signature. At the same time Witten proposed a description of elliptic genus as an index of certain Dirac-like operator on the loop space or more generally as the trace of certain operator associated with a conformal field theory (cf. [18]). Also at the

same time Witten proposed a generalization of elliptic genus for almost complex manifolds with first Chern class divisible by N (cf. [18]). A construction of such generalization was also given independently by F.Hirzebruch (cf. [13]). These genera are modular forms for $\Gamma_1(N)$. Motivated by attempts to extend rigidity property of elliptic genera, Krichever (cf. [17]) introduced an extension of Witten-Hirzebruch genus. Values of Krichever's elliptic genus are certain functions of several variables.

Also, in the physics literature there was proposed a version of elliptic genus such that its values on Calabi-Yau manifolds are (weak) Jacobi forms, i.e. certain functions on $H \times \mathbf{C}$. Several interesting properties were observed (cf. [8, 9, 16]). One of the most striking is the calculation of the elliptic genus on symmetric powers of a manifold M is terms of the elliptic genus of M itself (cf. [8]). Also several comments were made about possible relationship with mirror symmetry.

The purpose of this paper is to address from the mathematical point of view the properties of elliptic genera of Calabi-Yau manifolds and to study elliptic genera of toric varieties. We start by reviewing in Section 2 some aspects of previous work by Landweber-Stong, Ochanine, Witten, Hirzebruch and Krichever mentioned above. Then in Section 3 we introduce elliptic genus for Calabi-Yau manifolds, following suggestions from physics literature as the holomorphic Euler characteristic of the bundle:

$$y^{-\frac{dimM}{2}} \otimes_{n \geq 1} (\Lambda_{-yq^{n-1}} \bar{T}_M \otimes \Lambda_{-y^{-1}q^n} T_M \otimes S_{q^n} \bar{T}_M \otimes S_{q^n} T_M)$$

While this Euler characteristic can be calculated for any manifold and yields a holomorphic function of $H \times \mathbf{C}$, we show that for Calabi-Yau manifolds this Euler characteristic is a Jacobi form of weight 0 and index $\frac{d}{2}$ where d is the dimension of the manifold. This is done by using an expression for the characteristic series for such genus via theta functions. Theta functions did appear already in connection with elliptic genus but for the proofs of rigidity properties (cf. [20]) . We also show how elliptic genera of differentiable manifolds and almost complex manifolds mentioned above are related to the elliptic genus considered in present paper in the case of Calabi-Yau manifolds.

The crucial question is whether elliptic genus of Calabi-Yau manifold can be expressed in terms of Hodge numbers, i.e. whether the behavior of the elliptic genus in mirror correspondence can be deduced from known behavior of the Hodge numbers. We show in this paper that the answer is "no" provided the dimension of the manifold is at least 12 (15 for odd dimension). The Jacobi property of elliptic genus is the key issue in the argument. It allows us to calculate dimensions of spaces of functions on $H \times \mathbf{C}$ which are the elliptic genera of Calabi-Yau manifolds either by interpreting Jacobi forms as sections of certain bundles on the compactification of the quotient of $H \times \mathbf{C}$ by the Jacobi group or by using calculations of the space of weak Jacobi forms due to Eichler-Zagier (cf. [10]). A consequence of this is it that the space of functions generated by elliptic genera of Calabi-Yau manifolds is a free algebra on three generators of degrees 1, 2 and 3. This also implies that the space of elliptic genera of manifolds of dimension greater than 13 (or is equal to 12) is too big and elliptic genera cannot just depend on Hodge numbers. All of this is accomplished in Section 4.

Next we consider the elliptic genera for toric varieties. Use of the torus action of the bundle in the definition of the elliptic genus yields a series representing the elliptic genus in terms of the defining fan. One can restrict this elliptic genus to $y = -1$ which yields a formula for the elliptic genus studied by Landweber-Stong-Ochanine and Witten. Comparison of these series with standard expressions in terms of Eisenstein series yield very interesting identities. For example for $M = \mathbf{P}^2$ (after some easy modifications) we obtain the following identity:

$$\sum_{m \geq 1, n \geq 1} \frac{q^{m+n}}{(1+q^n)(1+q^m)(1+q^{m+n})} = \sum_{r \geq 1} q^{2r} \sum_{k|r} k$$

One can prove this identity by elementary means (cf. Section 5) but huge class of identities corresponding to other toric manifolds is somewhat mysterious.

Finally, in the last two sections the relationship

$$Ell(X; y, q) = (-1)^d Ell(X^*; y, q) \qquad (*)$$

between elliptic genera of a d-dimensional Calabi-Yau hypersurface X in a Fano toric variety and its mirror X^* is derived. The proof relies heavily on the work [3] by the first author. There are several ingredients in it which we hope have independent interest. First, the starting point is an interpretation of elliptic genus as certain trace which is a reminiscence of original Witten's definition but the (super)trace here is calculated on the cohomology of the chiral de Rham complex studied in [21] and the work [3] by the first author. The material from [3] needed for the proofs here is reviewed and used in Section 6. Second, since the chiral de Rham complex was defined in [3] for Gorenstein toric varieties and Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties this trace formula allows to define elliptic genus for such singular varieties as well. Based on results from [3] transformation law for the elliptic genus under mirror correspondence is proven. Then in the Section 7 it is shown that the elliptic genus of a Calabi-Yau hypersurface with Gorenstein singularities is a weak Jacobi form. This, together with the transformation law yields the relation (*).

2 A short review of elliptic genus

A genus (resp. complex genus) with values in a $\mathbf{Q}$-algebra R with unit is a ring homomorphism from the oriented cobordism ring Ω_*^{S0} (resp. complex cobordism ring Ω_*^U) to R. According to [12] such homomorphisms are in one to one correspondence with the formal power series $Q(x)$ with coefficients in R satisfying $Q(0) = 1$. The genus $\Psi_Q(x)$ corresponding to a series $Q(x)$ can be described as follows. Let $c(X) = \prod(1 + x_i)$ be a formal factorization of the total Pontryagin (in complex case Chern class) of a manifold X. Then the genus $\Psi_q(X)$ is:

$$\prod Q(x_i)[X]$$

where $[X]$ is the fundamental class of X and $\prod Q(x_i)$ is written as a polynomial in symmetric functions in x_i i.e. Pontryagin (resp. Chern in complex case) classes of X.

If $Q(0) \neq 1$ but $Q(0) \neq 0$ then the above formula still produces a R-valued invariant of the manifold. This "non-normalized" genus $\Psi_Q(X)$ is related to the genus $\Psi_{\tilde{Q}}(X)$, corresponding to the series $\tilde{Q}(x) = \frac{Q(x)}{Q(0)}$ as follows: $\Psi_Q(X) = \Psi_{\tilde{Q}}(X) \cdot Q(0)^{\frac{1}{4}dim_{\mathbf{R}}(X)}$ (exponent is $dim_{\mathbf{C}}X$ in complex case).

Elliptic genus of an oriented manifold X can be defined as $\mathbf{Q}[[q]]$-valued genus corresponding to the series (cf. volume [18] and references there):

$$Q(x) = \frac{x/2}{\sinh(x/2)} \prod_{n=1}^{\infty} [\frac{(1-q^n)^2}{(1-q^n \mathrm{e}^x)(1-q^n \mathrm{e}^{-x})}]^{(-1)^n}$$

It can be described also as

$$\hat{A}(X) ch\{\frac{R(T)}{R(1)^{dimX}}\}[X]$$

Here T is the tangent bundle, $\hat{A}$ is the ($\mathbf{Q}$-valued) genus corresponding to the series $\frac{x/2}{\mathrm{sinh}x/2}$ and

$$R(T) = \otimes_{l>0, l\equiv 0(2)} S_{q^l}(T) \otimes_{l>0, l\equiv 1(2)} \Lambda_{q^l}(T)$$

where

$$S_q(V) = \Sigma S^n(V) q^n, \Lambda_q(V) = \Sigma \Lambda^n(V) q^n$$

are generating series for symmetric and exteriors powers of a bundle V.

Elliptic genus of an oriented manifold is a modular form on $\Gamma_0(2)$ with rational coefficients if $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ where τ is in the upper half plane H.

Hirzebruch ([13]) and Witten ([18]) defined genera of complex manifolds which are modular forms on:

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) | c \equiv 0(N), a \equiv d \equiv 1(N)\}$$

provided the first Chern class of the manifold satisfies $c_1 \equiv 0(N)$.

In terms of characteristic series these genera can be defined as follows. For $\tau \in H$

$$\alpha = 2\pi \mathrm{i}(\frac{k}{N}\tau + \frac{l}{N}) \neq 0$$

and

$$\Phi(x, \tau) = (1 - \mathrm{e}^{-x}) \prod_{n=1}^{\infty} \frac{(1-q^n \mathrm{e}^x)(1-q^n \mathrm{e}^{-x})}{(1-q^n)^2} \tag{1}$$

the characteristic series is:

$$Q_{HW}(x, \tau) = x \mathrm{e}^{-\frac{k}{N}x} \frac{\Phi(x-\alpha)}{\Phi(x)\Phi(-\alpha)}$$

Krichever ([17]) considered a genus with characteristic series

$$Q_K(x, z, \omega_1, \omega_2, \kappa) = x \mathrm{e}^{-\kappa x} \frac{\sigma_{\omega_1, \omega_2}(x-z)}{\sigma_{\omega_1, \omega_2}(x) \sigma_{\omega_1, \omega_2}(-z)} \mathrm{e}^{\zeta_{\omega_1, \omega_2}(z)x}$$

where $z, \kappa \in \mathbf{C}^*$, $\sigma_{\omega_1,\omega_2}(z)$ and $\zeta_{\omega_1,\omega_2}(z)$ are Weierstrass σ and ζ functions corresponding to the same lattice in $\mathbf{C}$ (cf. [5]). If the lattice is $L = 2\pi\mathrm{i}\tau\mathbf{Z} + 2\pi\mathrm{i}\mathbf{Z}$ then the series Q_K specializes into Q_{HW} for $z = \alpha$ and $\kappa = -\frac{2k}{N}\zeta(\pi\mathrm{i}\tau) - \frac{2l}{N}\zeta(\pi\mathrm{i}) + \zeta(z)$.

Krichever proved rigidity theorem for such genus i.e. showed that for SU-manifolds with S^1-action the corresponding equivariant genus is a multiple of a trivial character generalizing similar results for orientable and complex manifolds (cf. [4, 13]).

In recent preprint ([25]) Burt Totaro identified the image of the universal genus corresponding to the series Q_K as the quotient of the SU-cobordism ring by the equivalence relation generated by(classical) flops.

In [9],[8] the authors considered a genus for (almost) complex manifolds for which $c_1 = 0$ or equivalently the structure group of the tangent bundle can be reduced to the group SU. It can be defined either as a (super)trace of a certain operator (cf. Definition 7.1 below) or as

$$\chi(M, q, y) = \int_M ch(\mathcal{ELL}_{q,y})td(M) \tag{2}$$

where

$$\mathcal{ELL}_{q,y} = y^{-\frac{\dim M}{2}} \otimes_{n\geq 1} (\Lambda_{-yq^{n-1}}\bar{T}_M \otimes \Lambda_{-y^{-1}q^n}T_M \otimes S_{q^n}\bar{T}_M \otimes S_{q^n}T_M) \tag{3}$$

In the next section we show that for Calabi-Yau manifolds the above elliptic genus is a weak Jacobi form. While this result is stated in [16], there seems to be no proof of it anywhere in the literature. We also spell out the relationship between the genus (2),(3) and the genera corresponding to the series Q_{HW} and Q_K. As a consequence we see that for Calabi-Yau manifolds the above elliptic genus up to simple factors coincides with the genus corresponding to Q_K.

3 Elliptic genera of Calabi-Yau varieties as weak Jacobi forms

In this section we assume that M is a complex compact manifold of dimension d.

Let

$$\theta(z, \tau) = q^{\frac{1}{8}}(2\sin\pi z)\prod_{l=1}^{l=\infty}(1-q^l)\prod_{l=1}^{l=\infty}(1-q^l\mathrm{e}^{2\pi\mathrm{i}z})(1-q^l\mathrm{e}^{-2\pi\mathrm{i}z})$$

where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ is the Jacobi's theta function (cf. [5]). As a theta function with characteristic this is $\theta_{1,1}(z, \tau)$ (cf. [22]). We have

$$\theta(\frac{z}{\tau}, -\frac{1}{\tau}) = -\mathrm{i}\sqrt{\frac{\tau}{\mathrm{i}}}\mathrm{e}^{\frac{\pi\mathrm{i}z^2}{\tau}}\theta(z, \tau)$$

(cf. [5]).

Proposition 3.1 If $c(T_M) = \prod(1 + x_i)$ then

$$\chi(M, q, y) = \int_M ch(\mathcal{ELL}_{q,y}) td(M)$$

where

$$\mathcal{ELL}_{q,y} = y^{-\frac{d}{2}} \otimes_{n\geq 1} \Lambda_{-yq^{n-1}} \bar{T}_M \otimes \Lambda_{-y^{-1}q^n} T_M \otimes S_{q^n} \bar{T}_M \otimes S_{q^n} T_M)$$

is equal to the integral over M of the degree d term in the expansion of:

$$E(z, \tau, x_1, ..., x_r) = \prod_i x_i \frac{\theta(\frac{x_i}{2\pi \mathrm{i}} - z, \tau)}{\theta(\frac{x_i}{2\pi \mathrm{i}}, \tau)}$$

where $y = \mathrm{e}^{2\pi \mathrm{i} z}$.

Proof. Indeed, χ is the genus corresponding to the series:

$$y^{-\frac{1}{2}} \prod_{n=1}^{n=\infty} \frac{(1 - yq^{n-1}\mathrm{e}^{-x})(1 - y^{-1}q^n\mathrm{e}^x)}{(1 - q^n\mathrm{e}^{-x})(1 - q^n\mathrm{e}^x)} \cdot \frac{x}{1 - \mathrm{e}^{-x}} =$$

$$y^{-\frac{1}{2}} \prod_{n=1}^{n=\infty} \frac{(1 - yq^n\mathrm{e}^{-x})(1 - y^{-1}q^n\mathrm{e}^x)(1 - y\mathrm{e}^{-x})}{(1 - q^n\mathrm{e}^{-x})(1 - q^n\mathrm{e}^x)} \cdot \frac{x}{1 - \mathrm{e}^{-x}}.$$

Hence for M with factored Chern class we have the following expression for generating series for χ

$$\mathrm{e}^{-\pi \mathrm{i} dz} \prod_{n=1}^{n=\infty} \frac{(1 - \mathrm{e}^{2\pi \mathrm{i}(z - \frac{x_i}{2\pi \mathrm{i}})}q^n)(1 - \mathrm{e}^{-2\pi \mathrm{i}(z - \frac{x_i}{2\pi \mathrm{i}})}q^n)}{(1 - \mathrm{e}^{2\pi \mathrm{i}\frac{x_i}{2\pi \mathrm{i}}}q^n)(1 - \mathrm{e}^{-2\pi \mathrm{i}\frac{x_i}{2\pi \mathrm{i}}}q^n)} \cdot \frac{(1 - \mathrm{e}^{2\pi \mathrm{i} z - x_i})x_i}{(1 - \mathrm{e}^{-x_i})} =$$

$$\mathrm{e}^{-\pi \mathrm{i} dz} \prod_i \frac{\theta(z - \frac{x_i}{2\pi \mathrm{i}})\sin(-\pi \frac{x_i}{2\pi \mathrm{i}})}{\theta(-\frac{x_i}{2\pi i})\sin\pi(z - \frac{x_i}{2\pi \mathrm{i}})} \cdot \frac{(1 - \mathrm{e}^{2\pi \mathrm{i} z - x_i})x_i}{(1 - \mathrm{e}^{-x_i})} =$$

$$\mathrm{e}^{-\pi \mathrm{i} dz} \prod_i \frac{\theta(z - \frac{x_i}{2\pi \mathrm{i}})}{\theta(-\frac{x_i}{2\pi \mathrm{i}})} \prod_i \frac{(\mathrm{e}^{\frac{-x_i}{2}} - \mathrm{e}^{\frac{x_i}{2}})}{(\mathrm{e}^{\pi \mathrm{i} z - \frac{x_i}{2}} - \mathrm{e}^{-\pi \mathrm{i} z + \frac{x_i}{2}})} \cdot \frac{(1 - \mathrm{e}^{2\pi \mathrm{i} z - x_i})x_i}{(1 - \mathrm{e}^{-x_i})} =$$

$$\prod_i \frac{x_i \theta(z - \frac{x_i}{2\pi \mathrm{i}})}{\theta(-\frac{x_i}{2\pi \mathrm{i}})} = \prod_i \frac{x_i \theta(\frac{x_i}{2\pi \mathrm{i}} - z)}{\theta(\frac{x_i}{2\pi \mathrm{i}})}.$$

$\square$

Theorem 3.2 Function $\chi(M, z, \tau)$ is a (weak) Jacobi of weight 0 and index $d/2$.

Weak here implies that while it obeys the transformation laws of the Jacobi forms, it does not satisfy regularity conditions at the cusps. Rather, the only condition at the cusp is that q appears with nonnegative powers only. We refer to [10] for precise definitions. Also, when d is odd, the definition of the Jacobi form must be modified to allow a character ([16]).

Proof. The condition at the cusp clearly holds because $\mathcal{ELL}_{q,y}$ has no negative powers of q. To verify that $\chi(M, z, \tau)$ is a Jacobi form of weight 0 and index $d/2$ with

character it is enough to check the modular properties of $\chi(M,z,\tau)$ for generators of Jacobi group. Indeed, if M is Calabi-Yau then we have:

$$\chi(M,z,\tau+1)=\chi(M,z,\tau) \tag{4}$$

$$\chi(M,\frac{z}{\tau},-\frac{1}{\tau})=\mathrm{e}^{\frac{\pi\mathrm{i}dz^2}{\tau}}\chi(M,z,\tau) \tag{5}$$

$$\chi(M,z+\tau,\tau)=(-1)^d\mathrm{e}^{-\pi\mathrm{i}d(\tau+2z)}\chi(M,z,\tau) \tag{6}$$

$$\chi(M,z+1,\tau)=(-1)^d\chi(M,z,\tau) \tag{7}$$

(6) and (7) follows from the identities:

$$\theta(z+1,\tau)=-\theta(z,\tau),\quad \theta(z+\tau,\tau)=-\mathrm{e}^{-2\pi\mathrm{i}z-\pi\mathrm{i}\tau}\theta(z,\tau)$$

and (4) is obvious. Let

$$\prod x_i\frac{\theta(\frac{x_i}{2\pi\mathrm{i}}-z,\tau)}{\theta(\frac{x_i}{2\pi\mathrm{i}},\tau)}=\sum_{\mathbf{k}}Q_{\mathbf{k}}(z,\tau)\mathbf{x}^{\mathbf{k}}$$

where $\mathbf{x}$ is a product of powers of x_i and $\mathbf{k}$ is multiindex. Hence, replacing $\tau\rightarrow -\frac{1}{\tau}, x_i\rightarrow\frac{x_i}{\tau}$ and $z\rightarrow\frac{z}{\tau}$ we obtain:

$$\prod_i(\frac{x_i}{\tau})\frac{\theta(-\frac{z}{\tau}+\frac{x_i}{2\pi\mathrm{i}\tau},-\frac{1}{\tau})}{\theta(\frac{x_i}{2\pi\mathrm{i}\tau},-\frac{1}{\tau})}=\sum_{\mathbf{k}}Q_{\mathbf{k}}(\frac{z}{\tau},-\frac{1}{\tau})(\frac{\mathbf{x}}{\tau})^{\mathbf{k}} \tag{8}$$

Left hand side in this identity can be replaced by

$$\prod_i(\frac{x_i}{\tau})\frac{\mathrm{e}^{\pi\mathrm{i}\frac{(-z+\frac{x_i}{2\pi\mathrm{i}})^2}{\tau}}\theta(-z+\frac{x_i}{2\pi\mathrm{i}},\tau)}{\mathrm{e}^{\pi\mathrm{i}\frac{(\frac{x_i}{2\pi\mathrm{i}})^2}{\tau}}\theta(\frac{x_i}{2\pi\mathrm{i}})}=$$

$$(\frac{1}{\tau})^d\prod\mathrm{e}^{\frac{-zx_i}{\tau}}x_i\frac{\mathrm{e}^{\frac{\pi\mathrm{i}z^2}{\tau}}\theta(-z+\frac{x_i}{2\pi\mathrm{i}})}{\theta(\frac{x_i}{2\pi\mathrm{i}})} \tag{9}$$

For degree d terms in (8) and (9) we have

$$Q_d(\frac{z}{\tau},-\frac{1}{\tau})=\mathrm{e}^{\frac{\pi\mathrm{i}dz^2}{\tau}}Q_d(z,\tau)$$

which yields (5). $\square$

We shall conclude this section with a description of relationship between the genus considered in this section and genera corresponding to series Q_{HW} and Q_K.

Proposition 3.3 Let $K(z, \omega_1, \omega_2, k)(X)$ be the genus of an almost complex manifold X that corresponds to Q_K. Then for a smooth Calabi-Yau manifold X of dimension d

$$K(2\pi\mathrm{i}z, 2\pi\mathrm{i}, 2\pi\mathrm{i}\tau, k)(X) = Ell(z,\tau)(X) \cdot (-\frac{\theta'(0,\tau)}{2\pi\mathrm{i}\,\theta(z,\tau)})^d$$

where $\zeta_{\omega_1,\omega_2}$ is the Weierstrass ζ-function corresponding to the lattice ω_1, ω_2.

Proof. Since (cf. [5] p.60)

$$\sigma_{\omega_1,\omega_2}(u) = \theta(\frac{u}{\omega_1}, \tau) \cdot \frac{\omega_1}{\theta'(0,\tau)} \cdot \mathrm{e}^{\zeta_{\omega_1,\omega_2}(\omega_1/2)\cdot \frac{u^2}{\omega_1}}$$

where $\tau = \frac{\omega_2}{\omega_1}$ we have

$$Q_K(x, 2\pi\mathrm{i}z, 2\pi\mathrm{i}, 2\pi\mathrm{i}\tau, \kappa) = -x\frac{\theta(\frac{x}{2\pi\mathrm{i}} - z)}{\theta(\frac{x}{2\pi\mathrm{i}})\theta(z)} \cdot \frac{\theta'(0,\tau)}{2\pi\mathrm{i}} \cdot \mathrm{e}^{(-k+\zeta_{2\pi\mathrm{i},2\pi\mathrm{i}z}(2\pi\mathrm{i}z)-2z\zeta_{2\pi\mathrm{i},2\pi\mathrm{i}z}(\pi\mathrm{i}))x}$$

and the claim follows from $c_1(X) = 0$ and Proposition 3.1. □

Proposition 3.4 Let $Ell(X, z, \tau)$ be elliptic genus and $H(X, \alpha, \beta, N, \tau)$ be Hirzebruch elliptic genus of level N ($\alpha, \beta \in \mathbf{Z}$ not both divisible by N) for a d-dimensional complex Calabi-Yau manifold X. Then for $\omega = \frac{2\pi\mathrm{i}(\alpha\tau+\beta)}{N}$ one has:

$$\Psi_{HW}(M, \alpha, \beta, N, \tau) = Ell(M, \frac{\omega}{2\pi\mathrm{i}}, \tau) \cdot \theta(-\frac{\omega}{2\pi\mathrm{i}})^{-d} \cdot \eta^{3d}(\tau)$$

Proof. Using

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{l\geq 1}(1-q^l)$$

the function Φ given by (1) can be written as

$$\Phi(x,\tau) = (1-\mathrm{e}^{-x})q^{\frac{1}{8}} \frac{\prod_{n=1}^{n=\infty}(1-q^n\mathrm{e}^x)(1-q^n\mathrm{e}^{-x})(1-q^n)}{\eta^3} =$$

$$(1-\mathrm{e}^{-x})\frac{\theta(\frac{x}{2\pi\mathrm{i}},\tau)}{2\sin(\pi\frac{x}{2\pi\mathrm{i}})\eta^3(\tau)} = \mathrm{e}^{\frac{x}{2}}\frac{\theta(\frac{x}{2\pi\mathrm{i}},\tau)}{\eta^3(\tau)}$$

Therefore

$$Q_{HW} = \frac{x\Phi(x-\omega,\tau)}{\Phi(x,\tau)\Phi(-\omega,\tau)}\mathrm{e}^{-\frac{\alpha x}{N}} = \frac{x\mathrm{e}^{\frac{-\alpha x}{N}}\theta(\frac{x-\omega}{2\pi\mathrm{i}},\tau)\eta^3(\tau)}{\theta(\frac{x}{2\pi\mathrm{i}},\tau)\theta(-\frac{\omega}{2\pi i},\tau)}$$

and the claim follows from $c_1(X) = 0$ and Proposition 3.1. □

4 Spaces spanned by elliptic genera

0. Notations. $\Gamma_{k,l}(n), k, l \in \mathbf{Q}, n \in \mathbf{Z}$ is the semidirect product of $k\mathbf{Z} \oplus l\mathbf{Z}$ and $\Gamma(n)$ i.e. the collection of (M, X) where $M \in \Gamma(n)$ and $X = (x_1, x_2), x_1 \in k\mathbf{Z}, x_2 \in l\mathbf{Z}$. The product is given by $(M, X)(M', X') = (MM', XM' + X')$. This product is the only one for which $(\gamma_1 \cdot \gamma_2)(z, \tau) = \gamma_1(\gamma(z, \tau))$ with standard action of $\Gamma_{1,1}(1)$ on $H \times \mathbf{C}$) i.e. $(m, n, \gamma)(z, \tau) = (\frac{z+m\tau+n}{c\tau+d}, \gamma(\tau))$.
$\mathbf{P}^2_n$ is the projective plane blown up at n points.

1. A compactification of $H \times \mathbf{C}/\Gamma_{1,1}(1)$.

One can obtain a compactification for $H \times \mathbf{C}/\Gamma_{1,1}(1)$ from compactification of quotients corresponding to congruence subgroups which are particularly simple.

Let us consider the compactification of $H \times \mathbf{C}/\Gamma_{1,1}(2)$ (cf. [26]) as the projective plane blown up at four base points of a generic pencil of quadrics. The compactification of $H/\Gamma(2)$ is the base of the pencil of elliptic curves (universal elliptic curve with level 2 structure) such that its quotient induced by involution $z \to -z$ is just mentioned pencil of quadrics. Three cusps of $\Gamma(2)$ correspond to three singular elements of this pencil. The images of points of order 2 on the universal elliptic curve with level 2 structure in the pencil of quadrics form 4 sections of the pencil of quadrics which are four exceptional curves in $\mathbf{P}^2_4$. In particular each component of a singular member of the pencil intersects with two such sections, i.e. the components are in one to one correspondence with the pairs of distinct points of order 2.

Since $H \times \mathbf{C}/\Gamma_{1,1}(1) = (H \times \mathbf{C}/\Gamma_{1,1}(2))/(\Gamma_{1,1}(1)/\Gamma_{1,1}(2))$ and $\Gamma_{1,1}(1)/\Gamma_{1,1}(2) = SL_2(\mathbf{F}_2)$ a compactification of $H \times \mathbf{C}/\Gamma_{1,1}(1)$ is given by the quotient $\mathbf{P}^2_4/SL_2(\mathbf{F}_2)$ (*). The action of $\Gamma_{1,1}(1)/\Gamma_{1,1}(2) = SL_2(\mathbf{Z})/\Gamma(2) = SL_2(\mathbf{F}_2)$ on singular fibres of the pencil of quadrics is specified by the action of $SL_2(\mathbf{Z})$ on points of order 2 on elliptic curves and hence on the sections of the pencil of quadrics. One can equivariantly blow down three lines in singular fibers that do not intersect the section which corresponds to the zero of the universal elliptic curve. Then the surface is simply a blowup of $\mathbf{P}^2$ in a single point and the action of $SL_2(\mathbf{F}_2)$ comes from the symmetries of a regular triangle in the plane. The plane is blown up in the center of the triangle, and the fibers over the cusps are the proper preimages of the symmetry lines of this triangle. This description allows us to calculate the quotient easily. In particular, it has only one component over the cusp of $SL_2(\mathbf{Z})$ and is smooth in the neighborhood of the cusp. The cover $\mathbf{P}^2_1 \to \mathbf{P}^2_1/SL_2(\mathbf{F}_2)$ has ramification occurring along the curves which are the fibres over the cusps of $H/\Gamma(2)$ and the fibres F_i and F_ω of the fibration $\mathbf{P}^2_1 \to H/\bar{\Gamma}(2)$ over $SL_2(\mathbf{F}_2)$-orbit of respectively $i = \sqrt{-1}$ and $\omega(\omega^3 = 1)$. Direct calculation shows that the action of the generator of $\mathbf{Z}_2$ on fibre F_i has two fixed points near which it acts as $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and hence yielding in the quotient two A_1-singularities. On the other hand the action of the generator of $\mathbf{Z}_3$

(*) In fact it supports a natural action of a bigger group $\Gamma_{\frac{1}{2},\frac{1}{2}}(1)/\Gamma_{1,1}(2)$ which is a semidirect product of $\mathbf{F}_2^2$ and $SL_2(\mathbf{F}_2)$ which is isomorphic to symmetric group S_4.The action on $\mathbf{P}^2_4$ is induced by permutation of four blown up points. The action of $S_3 = SL_2(\mathbf{F}_2)$ is the action of the subgroup of S_4 fixing one of four sections of the pencil, i.e. the one corresponding to the the zero of universal elliptic curve.

on F_ω has two fixed points of different type. Near one of these points it acts as $\left(\begin{smallmatrix}\omega & 0\\ 0 & \omega\end{smallmatrix}\right)$. Hence the quotient has the singularity which can be resolved by single blow up yielding single exceptional curve with self-intersection -3. Near another point it acts as $\left(\begin{smallmatrix}\omega & 0\\ 0 & \omega^2\end{smallmatrix}\right)$, so the singularity is A_2.

Moreover, the quotient $\mathbf{P}^2_1/SL_2(\mathbf{F}_2)$ has a structure of a toric surface, which corresponds to the fan whose four one-dimensional faces are generated by $(1,1)$, $(1,-1)$, $(-1,-1)$ and $(-1,2)$. The structure of the fibration comes from the projection along $x=y$ line. Notice that the cusp is not torus-invariant. The meaning of this toric structure is not clear, but it is certainly very useful in calculations of dimensions of line bundles over this surface. The easiest way to notice this toric structure is to employ the the Satake compactification of the quotient of Siegel modular space of rank 2 by the whole $SP_4(\mathbf{Z})$. The compactification of $H\times\mathbf{C}/\Gamma_{1,1}(1)$ is the exceptional divisor of the blow-up along the infinity curve. The Satake compactification is known to be the weighted projective space with weights $(2,3,5,6)$ ([14]) and the infinity curve corresponds to $x_5=x_6=0$ for an appropriate choice of x_5 and x_6. Then the blow-up is easily calculated by means of toric geometry and the fiber is the toric surface above.

One can also obtain the compactification starting form compactification of $H\times\mathbf{C}/\Gamma_{1,1}(3)$. The latter is $\mathbf{P}^2_9$ and the pencil of elliptic curves is a linear pencil containing a cubic curve and its Hessian (Hesse pencil). Such a pencil has 4 singular fibres corresponding to four cusps of $\Gamma(3)$ and each singular fibre consists of three lines formed by sides of a triangle. Similarly to the level 2 case $H\times\mathbf{C}/\Gamma_{1,1}(1)=(H\times\mathbf{C}/\Gamma_{1,1}(3))/(\Gamma_{1,1}(1)/\Gamma_{1,1}(3))=H\times\mathbf{C}/SL_2(\mathbf{F}_3)$. The quotient of $\mathbf{P}^2_9$ can be found by identifying $SL_2(\mathbf{F}_3)$ with the subgroup of $PGL_3(\mathbf{C})$ which is the image of the quotient of the Shephard-Todd group $G_{25}\subset GL_3(\mathbf{C})$ yielding the same compactification as the one obtained in using universal level 2 curve. (**).

2. Dimensions of the spaces of weak Jacobi forms of zero weight.

Weak Jacobi forms can be identified with sections of line bundles. In particular, weak Jacobi forms of weight 0 and index k are sections of the bundle $k[D_{(-1,-1)}]$ where $[D_{(-1,-1)}]$ is the line bundle of the toric divisor which corresponds to the point $(-1,-1)$ in the above description of the compactification as a toric surface. For small k these dimensions are collected in the following table.

k	0	1	2	3	4	5	6
dim	1	1	2	3	4	5	7

(10)

Alternatively, these dimensions could be derived from the Theorem 9.3 of [10]. That theorem states that the ring of weak Jacobi forms with even weight, any index and no character is $M_*[a,b]$ where M_* is the ring of modular forms, a has weight -2 and index 1, while b has weight 0 and index 1. Since M_* is generated by e_4 and e_6 of weights 4 and 6 respectively, the basis of the space of weak Jacobi forms of index

(**) The full Hesse group of order 216 which is the the latter image of Shephard-Todd group acts from this point of view as $\Gamma_{\frac{1}{3},\frac{1}{3}}(1)/\Gamma_{1,1}(3)$ which is a semidirect product of $SL_{\mathbf{F}_3}$ and $\mathbf{F}_3^2$.

k and weight 0 consists of $e_4^{\alpha}e_6^{\beta}a^{\gamma}b^{\delta}$ where $\alpha, \beta, \gamma, \delta$ are nonnegative, and

$$\begin{cases} 4\alpha + 6\beta - 2\gamma = 0 \\ \delta + \gamma = k \end{cases}$$

Therefore, the dimension of the space of weak Jacobi forms with index k and weight 0 equals the number of integer solutions of the system of inequalities

$$\alpha \geq 0,\ \beta \geq 0,\ 2\alpha + 3\beta \leq k.$$

When k is big, the growth is quadratic, but for small $k \leq 5$ it is linear. This is responsible for the fact that elliptic genera of Calabi-Yau varieties of small dimension are determined solely by their Hodge numbers. The following proposition follows easily from the above description.

Proposition 4.1 The ring of weak Jacobi forms of weight zero is freely generated over $\mathbf{C}$ by three elements of degrees 1, 2 and 3 respectively.

3. Elliptic genera of Calabi-Yau varieties. We will now show that elliptic genera of Calabi-Yau varieties span the corresponding spaces of weak Jacobi forms.

Theorem 4.2 For any $d > 0$ elliptic genera of Calabi-Yau varieties of dimension d span the space of weak Jacobi forms of weight 0 and index $d/2$ (with a character if d is odd).

Proof. Let us first handle the case of even dimension. In view of Proposition 4.1, it is enough to show that elliptic genera of Calabi-Yau varieties of dimensions 2, 4 and 6 span the corresponding spaces. Moreover, the q^0 part of the elliptic genus of M can be written in terms of Hodge numbers

$$Ell(M, y, 0) = y^{-\mathrm{dim}M/2} \sum_{p,q} (-1)^{p+q} h^{p,q} y^p(M)$$

so it is enough to show that the spaces spanned by above polynomials for all M have expected dimension. When $\mathrm{dim}M = 2$ $K3$ surface provides the necessary example. For dimension 4, we can look at $K3 \times K3$ and sextic X_6 in $\mathbf{P}^5$. Finally, in dimension 6 we look at $K3 \times K3 \times K3$, $K3 \times X_6$ and $X_8 \subset \mathbf{P}^7$.

In the case of odd dimension d notice that the space of weak Jacobi forms of weight 0 and index $d/2$ and character is isomorphic to the space of weak Jacobi forms with of weight 0 and index $(d-3)/2$. Really, transformation properties imply that all Jacobi forms with character vanish at $z = 1/2$, $z = \tau/2$ and $z = (\tau + 1)/2$. On the other hand, there is a unique (up to a constant) weak Jacobi form with character of index 3/2 given by

$$f(z, \tau) = \frac{\theta(2z, \tau)}{\theta(z.\tau)}.$$

It appears in [23] in the description of the elliptic genera of Calabi-Yau threefolds. For any weak Jacobi form g of index $d/2$ the function g/f is holomorphic on $H \times \mathbf{C}$. Moreover, it is easy to see that it has a q-expansion, so it is a weak Jacobi form of index $(d-3)/2$ and no character.

As a result, to show that elliptic genera of Calabi-Yau varieties of odd dimension span the whole space of Jacobi forms it is enough to consider products of even-dimensional Calabi-Yau varieties with a quintic in $\mathbf{P}^4$ (or any other Calabi-Yau threefold with non-zero Euler characteristic). $\square$

4. Elliptic genera and Hodge numbers. The goal of this subsection is to discuss the relation between elliptic genera and Hodge numbers. As was already mentioned in the proof of Theorem 4.2,

$$Ell(M, y, 0) = y^{-\dim M/2} \sum_{p,q} (-1)^{p+q} h^{p,q} y^p(M)$$

so to know $Ell(y, 0)$ is to know $\chi^p = \sum (-1)^q h^{p,q}$ for all p. On the other hand, $Ell(y, q)$ certainly can't be used to determine individual $h^{p,q}$ even in dimension 2.

Theorem 4.3 If dimension of a Calabi-Yau manifold is less than 12 or is equal to 13, then the numbers χ_p determine its elliptic genus uniquely. In all other dimensions there exist Calabi-Yau manifolds with the same $\{\chi_p\}$ but distinct elliptic genera. Here we allow disconnected Calabi-Yau manifolds.

Proof. For M of even dimension d there are at most $d/2$ independent numbers χ_p. Really, because of the symmetry of Hodge diagram we only need to consider $\chi_0, ..., \chi_{d/2}$ and there is also an additional linear relation, see [19]. For M of odd dimension d there are $(d-3)/2$ independent numbers χ_p, because we also have $\chi_0 = 0$.

There is a ring homomorphism from the ring spanned by all elliptic genera to the ring spanned by all $Ell(M, y, 0)$. The table (10) shows that this homomorphism can not be surjective in even dimensions starting at 12 and in odd dimensions starting at 15. It is easy to see that some polynomial in the generators of the ring of Jacobi forms (constructed from $K3$, X_6 and X_8) with integer coefficients goes to zero when restricted to $q = 0$. Then one collects positive and negative terms and interprets them as elliptic genera of disjoint unions of products of $K3$, X_6 and X_8. This proves the second part of the theorem. To prove the first part, all we need to show is that the spaces spanned by $\{\chi_p\}$ have dimension $d/2$ for $d = 2, 4, 6, 8, 10$, which is an explicit calculation. $\square$

Remark 4.4 We have first conjectured the above result after extensive Mathematica calculations of elliptic genera and Hodge numbers in terms of Chern classes.

5 Elliptic genera of toric varieties

The goal of this section is to calculate elliptic genus $Ell(y, q)$ of an arbitrary complete smooth toric variety. As a byproduct, we notice a curious identity and give a direct

elementary proof of it. One can also define elliptic genus of an arbitrary complete toric variety with only Gorenstein singularities but we delay it until Section 7 when the required techniques are developed.

Let us first describe the data that define a complete toric variety of dimension d, see [7, 11, 24]. We have a lattice (which here simply means a free abelian group) M of rank d, its dual lattice N and a complete polyhedral fan Σ in N. If all cones $C^* \in \Sigma$ of maximum dimension are simplicial and are generated by a basis of N, then the corresponding variety $\mathbf{P}_\Sigma$ is smooth, and vice versa. We will denote this variety simply by $\mathbf{P}$.

We have observed in Section 3 that elliptic genus $Ell(y, q)$ of any smooth variety could be calculated as an Euler characteristics of a certain double graded sheaf. Here we change our viewpoint a little bit and work with supersheaves. For the purposes of this section, supersheaves are simply usual sheaves with $\mathbf{Z}/2\mathbf{Z}$ grading. The corresponding graded components are called even and odd.

Proposition 5.1 Let X be a smooth complete variety of dimension d. Consider the following double graded supersheaf on it which we still denote by $\mathcal{ELL}$ abusing notations slightly.

$$\mathcal{ELL}(X) = \otimes_{k\geq 1} \wedge (yq^{k-1}T^*) \otimes_{k\geq 1} \wedge (y^{-1}q^kT) \otimes_{k\geq 1} \mathrm{Sym}(q^kT^*) \otimes_{k\geq 1} \mathrm{Sym}(q^kT),$$

where the grading is given by the powers of y and q and the parity is given by the parity of the degree of y. Then $Ell(X; y, q)$ is the $y^{-d/2}$ times the super Euler characteristics of the above sheaf. As usual, super Euler characteristics means the Euler characteristics of the even part minus the Euler characteristics of the odd part.

Proof. This result follows directly from Hirzebruch-Riemann-Roch Theorem and definition of $Ell(y, q)$. $\square$

When $X = \mathbf{P}$ is a toric variety, we get some help from the action of the torus $(\mathbf{C}^*)^d$. The idea is to somehow split the calculation into the sum over the characters $m \in M$. Cohomology of $\mathcal{ELL}(\mathbf{P})$ could be calculated as Čech cohomology for the open affine covering $\{\mathbf{A}_C = \mathrm{Spec}(\mathbf{C}[C])\}$ defined by cones $C^* \in \Sigma$. Intersection of any number of such subsets is another open subset of this type, so the covering is acyclic for $\mathcal{ELL}(\mathbf{P})$. For fixed powers of y and q the cohomology of the corresponding graded part of the Čech complex is finite-dimensional, but the entries of the Čech complex itself have, of course, infinite dimension. Fortunately, the action of $(\mathbf{C}^*)^d$ naturally extends to $\mathcal{ELL}(\mathbf{P})$, so sections of $\mathcal{ELL}(\mathbf{P})$ over any such affine subset $\mathbf{A}_C$ admit natural grading by the lattice M, which is compatible with Čech differential. The important remark is that for a fixed C, m and powers of y and q, the corresponding graded part of the Čech complex is finite-dimensional. Thus, to calculate the Euler characteristics of $\mathcal{ELL}(\mathbf{P})$, one can simply calculate the alternating sum of the dimensions of these spaces for a fixed m and then add the results for all m. This gives us

$$Ell(\mathbf{P}; y, q) = y^{-d/2} \sum_{m\in M} (\sum_{C_0^*,\ldots,C_k^*} (-1)^k \mathrm{dim}_m H^0(\mathbf{A}_{C_0} \cap \ldots \cap \mathbf{A}_{C_k}, \mathcal{ELL}(\mathbf{P})))$$

where dim is understood in a super sense (dimension of even minus dimension of odd). It is convenient to account for all m simultaneously by introducing a multivariable t and using $\sum_{m\in M} t^m \mathrm{dim}_m$. In the end we will put $t=1$, but for now we will keep it to assure convergence of all expressions.

Let us first consider $\mathbf{A}_C$ with C of maximum dimension. It is isomorphic to the affine space of dimension d with coordinates $x^{m_1},...,x^{m_d}$ where $m_1,...,m_d$ generate the cone C and form the basis of M. Sections of $\mathcal{ELL}(\mathbf{P})$ over this affine set could be easily calculated. Really, it is a superpolynomial ring (polynomial ring of even variables tensored with the exterior algebra of the space of odd variables) with even variables $x^{m_i}, d(x^{m_i})q^k, k\geq 1, (\partial/\partial x^{m_i})q^{k-1}, k\geq 1$ and odd variables $d(x^{m_i})q^k y, k\geq 1, (\partial/\partial x^{m_i})q^{k-1}y, k\geq 1$. Therefore,

$$\sum_{m\in M} t^m \mathrm{dim}_m H^0(\mathbf{A}_C,\mathcal{ELL}(\mathbf{P})) = \prod_{i=1,...,d}\prod_{k\geq 1}\frac{(1-t^{m_i}yq^{k-1})(1-t^{-m_i}y^{-1}q^k)}{(1-t^{m_i}q^{k-1})(1-t^{-m_i}q^k).}$$

Unfortunately, in this form it is hard to combine together information from several cones. We are helped by the following observation.

Proposition 5.2

$$\prod_{i=1,...,d}\prod_{k\geq 1}\frac{(1-t^{m_i}yq^{k-1})(1-t^{-m_i}y^{-1}q^k)}{(1-t^{m_i}q^{k-1})(1-t^{-m_i}q^k)} = \sum_{m\in M} t^m \prod_{i=1,...,d}\left(\frac{1}{1-yq^{m\cdot n_i}}\right) G(y,q)^d$$

where

$$G(y,q)=\prod_{k\geq 1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}$$

and n_i are generators of C^*.

Proof. It is sufficient to show this for $d=1$, that is we need to show that

$$\prod_{k\geq 1}\frac{(1-tyq^{k-1})(1-t^{-1}y^{-1}q^k)}{(1-tq^{k-1})(1-t^{-1}q^k)} = \sum_{m\in\mathbf{Z}}\left(\frac{t^m}{1-yq^m}\right)\prod_{k\geq 1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}. \tag{11}$$

We are aware of two proofs of this identity. The first one uses some representation theory and is based on the calculation of cohomology of a certain explicit operator on the Fock space of two free bosons and two free fermions. It is one of the main ingredients of the preprint [3]. There also exists an elementary proof which is sketched below. If we treat t and q as complex numbers with $|q|<|t|<1$ then both sides of the equation are meromorphic functions of z (recall that $y=\mathrm{e}^{2\pi\mathrm{i}z}$). It is straightforward to show that the ratio of the two sides is double periodic, because both sides acquire a factor of $-t^{-1}y^{-1}$ under $y\to yq$. Moreover, the ratio

$$\sum_{m\in\mathbf{Z}}\left(\frac{t^m}{1-yq^m}\right)\prod_{k\geq 1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)(1-tq^{k-1})(1-t^{-1}q^k)}{(1-q^k)^2(1-tyq^{k-1})(1-t^{-1}y^{-1}q^k)}$$

has no poles. Really, because of the periodicity, it is enough to check that there is no pole at $y = t^{-1}$, which follows from

$$\sum_{m\in\mathbf{Z}} \frac{t^m}{1-t^{-1}q^m} = \sum_{m>0}\sum_{k\geq 0} t^m t^{-k} q^{mk} + \frac{1}{1-t^{-1}} + \sum_{m>0}\sum_{k\geq 1} (-1) t^{-m} t^k q^{mk}$$

$$= \sum_{m>0} t^m + \frac{1}{1-t^{-1}} + \sum_{n>0} q^n \sum_{mk=n,m,k>0} (t^{m-k} - t^{k-m}) = 0.$$

Then it remains to observe that at $y = 1$ both sides are equal to 1.

Remark 5.3 Expressions $\frac{1}{1-yq^{m\cdot n_i}}$ in the above formulas could be understood as power series in q whose coefficients are rational functions in y, that is when $m\cdot n_i < 0$, they still have to be expanded around $q = 0$. Basically, when you multiply it by $G(y,q)^d$ as we do, the denominators disappear and the resulting functions should be expanded around $q = 0$, so we might as well expand the original expressions around $q = 0$.

One can also show that the same formula holds for sections of $\mathcal{ELL}(\mathbf{P})$ over $\mathbf{A}_C$ where C^* is allowed to have a dimension smaller than d.

Proposition 5.4 For any cone C^* in Σ we have

$$\sum_{m\in M} t^m \mathrm{dim}_m H^0(\mathbf{A}_C, \mathcal{ELL}(\mathbf{P})) = \sum_{m\in M} t^m \prod_{i=1,\ldots,\mathrm{dim}C^*} \left(\frac{1}{1-yq^{m\cdot n_i}}\right) G(y,q)^d$$

where

$$G(y,q) = \prod_{k\geq 1} \frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}$$

and n_i are generators of C^*.

Proof. The corresponding $\mathbf{A}_C$ could be (non-canonically) split as $\mathbf{C}^{\mathrm{dim}C^*} \times (\mathbf{C} - \{0\})^{d-\mathrm{dim}C^*}$. Then over the torus we can use $\frac{dx}{x}$ and $x\partial_x$ as a basis for differential forms and vector fields. They have zero grading, and it is then easy to calculate the contribution of a torus. Combined with the previous proposition for $\mathbf{C}^{\mathrm{dim}C^*}$, this gives the desired formula. □

We are now in a position to formulate the main result of this section.

Theorem 5.5 Let $\mathbf{P}$ be a smooth toric variety given by the fan Σ. Then

$$Ell(\mathbf{P}, y, q) = y^{-d/2} \sum_{m\in M} \sum_{C^*\in\Sigma} (-1)^{\mathrm{codim}C^*} \left(\prod_{i=1,\ldots,\mathrm{dim}C^*} \frac{1}{1-yq^{m\cdot n_i}}\right) G(y,q)^d$$

where

$$G(y,q) = \prod_{k\geq 1} \frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}$$

and n_i are generators of C^*.

Proof. Because of Proposition 5.4 it is enough to show that in the alternating sum over the Čech complex each cone $C^* \in \Sigma$ is counted with the coefficient $(-1)^{\mathrm{codim} C^*}$. We will carry an induction on codimension of C^*. Obviously, the cones that do not contain C^* could be ignored. Then we can quotient out the subspace generated by C^*. The cones that contain C^* will form a complete fan on the remaining subspace of dimension codimC^*. The sum of $(-1)^k$ over all possible intersections of $k+1$ cones in this fan is, of course, 1 by binomial formula. On the other hand this sum splits according to possible intersections of the cones in the fan. All cones, except the vertex (image of C^*) contribute $(-1)^{\mathrm{codim}}$ by the induction assumption. So the occurrence of C^* is

$$1 - \sum_{C_1*, C^* \subset C_1^*} (-1)^{\mathrm{codim} C_1^*}$$

and the above sum could be expressed as the Euler characteristics of a sphere of dimension codim$C^* - 1$, which finishes the proof. □

Remark 5.6 Alternatively, one could use a slight modification of the Čech complex where you use each cone once and the differential consists of the restriction maps with the signs as in the singular homology complex of Σ. Then the above formula is automatic, but the difficult part is to show that the complex can be used to calculate the cohomology of any coherent sheaf.

Since a complete toric variety can never be a Calabi-Yau, we are mostly interested here in the $y = -1$ case. Also, one can take $G(-1,q)^d$ outside of the summation sum once we agree to expand around $q = 0$, see Remark 5.3. If we denote

$$\widehat{Ell}(X;q) = (-1)^{d/2} Ell(X;-1,q) G(-1,q)^{-d}$$

we get the following result.

Theorem 5.7 If $\mathbf{P}$ is a smooth complete toric variety then

$$\widehat{Ell}(\mathbf{P};q) = \sum_{m \in M} \left(\sum_{C^* \in \Sigma} (-1)^{\mathrm{codim} C^*} \prod_{i=1,\dots,\mathrm{dim} C^*} \frac{1}{1+q^{m \cdot n_i}} \right).$$

The inner sum here is taken first. Then for a given degree of q only a finite number of m contribute to $\widehat{Ell}$.

Proof. The only thing to prove here is the last statement. However, the part of the sheaf $\mathcal{ELL}(\mathbf{P})$ for a fixed degree of q is coherent, so its cohomology has finite dimension. Therefore, only a finite number of m contribute to $Ell(\mathbf{P},-1,q)$ at a given degree. Multiplication by $G(-1,q)^{-d}$ does not change this. □

Remark 5.8 Notice that $\widehat{Ell}(\mathbf{P})$ is precisely the Landweber-Stong genus, see [18]. In particular, it is known to be a modular form with respect to the index three subgroup $\Gamma_0(2)$ in the full modular group. However, we do not see it from the formulas above. One may also wonder if these formulas could be modified to give examples of modular forms at different levels. An obvious thing to try is to use $\sum_{m \in M'}$ where M' is a sublattice of M.

Remark 5.9 We can compare our result for $\mathbf{P} = \mathbf{P}^2$ with the known Landweber-Stong genus of $\mathbf{P}^2$. After rather easy simplifications, this results in the following curious identity.

$$\sum_{m\geq 1,n\geq 1} \frac{q^{m+n}}{(1+q^m)(1+q^n)(1+q^{m+n})} = \sum_{r\geq 1} q^{2r} \sum_{k|r} k.$$

Even though our discussion above provides another proof of it, there exists a direct derivation. Also, George Andrews have shown us how to reduce this identity to the result of [1].

Proof of the $\mathbf{P}^2$ *identity.* It is easy to show that

$$\frac{xy}{(1+x)(1+y)(1+xy)} = \sum_{a\geq 1,b\geq 1,\min(a,b)=odd} x^a y^b (-1)^{a+b},$$

which implies that

$$\sum_{m\geq 1,n\geq 1} \frac{q^{m+n}}{(1+q^m)(1+q^n)(1+q^{m+n})} = \sum_{d\geq 2} q^d \sum_{a\geq 1,b\geq 1,m\geq 1,n\geq 1,ma+nb=d,\min(a,b)=odd} (-1)^{a+b}.$$

For a given d we look at the set of all solutions to $a \geq 1, b \geq 1, m \geq 1, n \geq 1, ma + nb = d, \min(a,b) = odd$ which we denote by I. We denote by J the set obtained from I by excluding all solutions with $m = n$ and $a + b = even$. We will show that the set J does not contribute to the above sum. We denote by J_{odd} and J_{even} the parts of J with odd and even $a+b$ respectively and plan to show that there is a one-to-one correspondence between these two sets.

Given an element (a,b,m,n) in J_{even} we construct an element in J_{odd} either as $(a+b,b,m,n-m)$ or $(a,a+b,m-n,n)$ depending on which of m and n is bigger (they can not be equal, because we are in J). Given an element (a,b,m,n) in J_{odd} we construct an element in J_{even} as $(a-b,b,m,m+n)$ or $(a,b-a,m+n,n)$ depending on which of a and b is bigger. One can show that these two maps are well-defined and are inverses of each other.

So we have

$$\sum_{m\geq 1,n\geq 1} \frac{q^{m+n}}{(1+q^m)(1+q^n)(1+q^{m+n})} = \sum_{d\geq 2} q^d \sum_{a\geq 1,b\geq 1,m\geq 1,m(a+b)=d,\min(a,b)=odd,a+b=even} 1$$

which is easily seen to equal

$$\sum_{r\geq 1} q^{2r} \sum_{k|r} k. \qquad \square$$

6 Elliptic genera and mirror symmetry

The goal of this section is to show that elliptic genera of two mirror symmetric hypersurfaces in toric varieties coincide up to an expected sign. Because of Theorem 4.3, this provides a new check of mirror symmetry in high dimension. We introduce a concept of elliptic genus for hypersurfaces in toric Fano varieties with Gorenstein

singularities, and prove the mirror duality for these genera as well. Unfortunately, the argument uses full force of the machinery of [3] and [21], even in the smooth case. We will be reviewing briefly necessary techniques as we go on.

Chiral de Rham complex $\mathcal{MSV}(X)$ of any smooth variety X was defined in [21]. It is a sheaf of vector spaces over X equipped with a double grading by eigenvalues of $J[0]$ and $L[0]$. It is also given a $\mathbf{Z}/2\mathbf{Z}$ (even-odd) grading induced from the Z-grading by $J[0]$. We must mention that $\mathcal{MSV}(X)$ itself in not a quasi-coherent sheaf. The "multiplication by functions" map of sheaves $\mathcal{O}(X)\times\mathcal{MSV}(X)\to\mathcal{MSV}(\mathcal{X})$ is defined but is not associative. However, $\mathcal{MSV}(X)$ possesses a natural filtration compatible with the grading and the above multiplication such that the graded object is a quasi-coherent sheaf isomorphic to

$$\mathcal{ELL}=\otimes_{k\geq 1}\wedge(yq^{k-1}T^*)\otimes_{k\geq 1}\wedge(y^{-1}q^kT)\otimes_{k\geq 1}\mathrm{Sym}(q^kT^*)\otimes_{k\geq 1}\mathrm{Sym}(q^kT),$$

see [21]. Here the powers of y and q stand for the eigenvalues of $J[0]$ and $L[0]$ respectively. Since at each power of q in the above expression we have a coherent sheaf (this means that $\mathcal{MSV}$ is loop-coherent, or loco, in the terminology of [3]), the cohomology of $\mathcal{MSV}(X)$ are finite-dimensional vector spaces over $\mathbf{C}$ at every given eigenvalue of $L[0]$. As a result, the (double graded, super) Euler characteristics of $\mathcal{MSV}(X)$ is well-defined and coincides with the Euler characteristics of the sheaf above. Together with Proposition 5.1, this prompts the following definition.

Definition 6.1 Let X be any variety of dimension d for which there is defined the chiral de Rham complex $\mathcal{MSV}(X)$. Then elliptic genus $Ell(X;y,q)$ is defined by

$$Ell(X;y,q)=y^{-d/2}\mathrm{SuperTrace}_{H^*(\mathcal{MSV}(X))}y^{J[0]}q^{L[0]}.$$

This allows us to extend the notion of elliptic genus to some singular varieties for which chiral de Rham complexes have been constructed. At this point, such varieties include arbitrary Gorenstein toric varieties and Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties, see [3]. The above formula makes sense because it was shown in [3] that for fixed $(J[0],L[0])$ eigenvalues the corresponding eigenspace in H^* is finite-dimensional. We have shown in Section 3 that for any smooth Calabi-Yau variety thus defined elliptic genus is a weak Jacobi form.

We will now describe how to calculate elliptic genera of Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties in terms of combinatorial structures that define them. We recall that such a family of Calabi-Yau hypersurfaces is determined by the following data, see [2].

Let M_1 and N_1 be dual free abelian groups of rank $d+1$. Denote by M and N two dual free abelian groups such that $M=M_1\oplus\mathbf{Z}$ and $N=N_1\oplus\mathbf{Z}$. Element $(0,1)\in M$ is denoted by deg and element $(0,1)\in N$ is denoted by $\deg^*$. There are dual reflexive polytopes $\Delta\in M_1$ and $\Delta^*\in N_1$ which give rise to dual cones $K\subset M$ and $K^*\subset N$. Namely, K is a cone over $(\Delta,1)$ with vertex at $(0,0)_M$, and similarly for K^*. There is a complete fan on N_1 whose one-dimensional cones are generated

by some lattice points in Δ^* (in particular, by all vertices). This fan induces the decomposition of the cone K^* into subcones, each of which includes $\deg^*$. Let us denote this decomposition by Σ. It will later turn out that elliptic genus does not depend on the choice of Σ (that is, partial crepant toric desingularizations do not alter it).

The formula for the elliptic genus that is the cornerstone of the argument is given by the following proposition.

Proposition 6.2 Let X be a generic hypersurface in the Gorenstein toric Fano variety defined by the combinatorial data above. Then

$$Ell(X; y, q) = y^{-d/2} \sum_{m \in M} \left(\sum_{n \in K^*} y^{n \cdot \deg - m \cdot \deg^*} q^{m \cdot n + m \cdot \deg^*} G(y, q)^{d+2} \right)$$

where

$$G(y, q) = \prod_{k \geq 1} \frac{(1 - yq^{k-1})(1 - y^{-1}q^k)}{(1 - q^k)^2}.$$

The inner summation times G^{d+2} is a well-defined double series in y and q, when G is expanded around $q = 0$. Then the outer sum turns out to make sense as a double power series in y and q, that is only a finite number of m contribute to a given coefficient.

Remark 6.3 This formula clearly shows that elliptic genus does not depend on the choice of desingularization. In particular, if X admits a crepant toric desingularization $\hat{X}$, then

$$Ell(X; y, q) = Ell(\hat{X}; y, q).$$

Before we begin to prove Proposition 6.2, we need to recall more background material from [3]. That paper contains the calculation of the cohomology of the chiral de Rham complex for toric varieties and hypersurfaces in them. If the ambient toric variety is smooth, then $H^*(\mathcal{MSV})$ of the hypersurface can be calculated as the cohomology of a certain vector space by a certain operator, both described below.

This space is built from tensor products of irreducible modules of infinite Heisenberg and Clifford algebra. Those are the affinization (resp. Clifford affinization, cf. [15] p.26) of abelian Lie algebra $M \oplus N$ in which all elements are considered even (resp. odd) with supersymmetric (resp. skew supersymmetric) bilinear form given by natural pairing $M \times N \to \mathbf{Z}$. In other words these algebras (resp. $H(M \oplus N)$ and $Cl(M \oplus N)$) are

$$(M \oplus N) \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}K$$

with commutator

$$\{x \otimes t^k, y \otimes t^l\}_- = k(x, y)\delta_{k+l,0}K$$

resp.

$$\{x \otimes t^k, y \otimes t^l\}_+ = (x, y)\delta_{k+l,0}K$$

(here $\{.,.\}_-$ and $\{.,.\}_+$) denote commutator and anticommutator respectively). We will always assume that the commutator K acts as identity and we identify K for both algebras. We will thus consider

$$\mathbf{g} = (M \oplus N)_H \otimes \mathbf{C}[t, t^{-1}] \oplus (M \oplus N)_{Cl} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}K$$

where subscripts are used to distinguish between two copies of the lattice.

The "canonical representations" of $\mathbf{g}$ (the Fock spaces) denoted $\mathrm{Fock}_{m_0 \oplus n_0}$ can be obtained as follows. We define $b\mathbf{g}$ as

$$b\mathbf{g} = bH(M \oplus N) \oplus bCl(M \oplus N) \oplus \mathbf{C}K$$

$$= \oplus_{k \geq 0}(M \oplus N)_H \otimes t^k \oplus (\oplus_{k>0} M_{Cl} \otimes t^k) \oplus (\oplus_{k \geq 0} N_{Cl} \otimes t^k) \oplus \mathbf{C}K$$

and view $\mathbf{C}$ as a $b\mathbf{g}$-module such that

$$bCl(M \oplus N) \cdot 1 = 0, K \cdot 1 = 1, x_H \otimes t^0 \cdot 1 = x \cdot (m_0 \oplus n_0),$$

$$x_H \otimes t^k \cdot 1 = 0 \text{ for } k \geq 1 \ (x \in M \oplus N).$$

Then we put

$$\mathrm{Fock}_{m_0 \oplus n_0} = U(\mathbf{g}) \otimes_{b\mathbf{g}} \mathbf{C}.$$

We will be using the notations of [3]. There, algebra $\mathbf{g}$ is described in terms of generators $n \cdot A[k], m \cdot B[k], m \cdot \Phi[k], n \cdot \Psi[k]$ for all $k \in \mathbf{Z}$ where $n \cdot A[k]$ stands for $(0 \oplus n)_H \otimes t^k$ and similarly for B, Φ, Ψ. The only non-trivial super-commutators are

$$\{m \cdot B[k], n \cdot A[l]\}_- = (m \cdot n) k \delta^0_{k+l}, \ \{m \cdot \Phi[k], n \cdot \Psi[l]\}_+ = (m \cdot n) \delta^0_{k+l}.$$

As a a vector space $\mathrm{Fock}_{m_0 \oplus n_0}$ is a tensor product of the symmetric algebra of a certain infinite-dimensional space and an exterior algebra of another infinite-dimensional space. The first space is

$$\oplus_{k \geq 1} M \cdot B[-k]\mathbf{C} \oplus_{k \geq 1} N \cdot A[-k]\mathbf{C}$$

and the second space is

$$\oplus_{k \geq 1} M \cdot \Phi[-k]\mathbf{C} \oplus_{k \geq 0} N \cdot \Psi[-k]\mathbf{C}.$$

Here $M \cdot B[-k]\mathbf{C}$ means simply a copy of $M_{\mathbf{C}}$, one for each k, and similarly for A, Φ and Ψ.

Elements of $\mathrm{Fock}_{m \oplus n}$ can be written as

$$\begin{gathered}\sum (m_{11} \cdot B[-1])(m_{12} \cdot B[-1])...(m_{21} \cdot B[-2])(m_{22} \cdot B[-2])... \\ (n_{11} \cdot A[-1])(n_{12} \cdot A[-1])...(n_{21} \cdot A[-2])(n_{22} \cdot A[-2])... \\ (m'_{11} \cdot \Phi[-1])(m'_{12} \cdot \Phi[-1])...(m'_{21} \cdot \Phi[-2])(m'_{22} \cdot \Phi[-2])... \\ (n'_{11} \cdot \Psi[0])(n'_{12} \cdot \Psi[0])...(n'_{21} \cdot \Psi[-1])(n'_{22} \cdot \Psi[-1])...|m, n > \end{gathered} \tag{12}$$

where there are only finitely many summands and finitely many factors in each of the summands. It should be understood that the above expressions are linear in m_{ij} and n_{ij}, and A, B, Φ and Ψ super-commute. Commutation rule is necessary to understand the relation to the product of symmetric and exterior powers.

Operators $m\cdot B[k], k<0$, $n\cdot A[k], k<0$, $m\cdot\Phi[k], k\le 0$ and $n\cdot\Psi[k], k<0$ act on this space via multiplication in the (super)-polynomial rings. Operators $n\cdot A[0]$ and $m\cdot B[0]$ act as scalars $m_0\cdot n$ and $m\cdot n_0$ respectively. Other operators act as derivations on these rings. Our notations are set up in a way that $A[k]$, $B[k]$, $\Phi[k]$ and $\Psi[k]$ are vector-valued operators on this Fock space.

For any subset of $M\oplus N$ we can consider the direct sum of the corresponding Fock spaces. For example, we will consider

$$\mathrm{Fock}_{M\oplus N}=\oplus_{m_0\in M,n_0\in N}\mathrm{Fock}_{m_0\oplus n_0}$$

and

$$\mathrm{Fock}_{M\oplus K^*}=\oplus_{m_0\in M,n_0\in K^*}\mathrm{Fock}_{m_0\oplus n_0}.$$

Out of the operators $m\cdot B[l]$, $n\cdot A[l]$, $m\cdot\Phi[l]$ and $n\cdot\Psi[l]$ one can construct the operators $L[0]$ and $J[0]$ which act on each of the spaces $\mathrm{Fock}_{m_0\oplus n_0}$. The details could be found in [3], but it is useful to mention that the "monomials" in the polynomial ring written above are eigen-vectors of these operators. The eigen-values are calculated as follows.

$L[0]$ *counts the opposite of the sum of indices [ ] in the above expression (12) of the element, plus* $m\cdot n+\deg^*\cdot m$*;*

$J[0]$ *counts the number of occurrences of* Φ *minus the number of occurrences of* Ψ *plus* $\deg\cdot n-\deg^*\cdot m$*.*

In addition to these operators there are defined *vertex operators* $\mathrm{e}^{\int m\cdot A+n\cdot B}[l]$ for all m and n in the corresponding lattices and all integer l. These operators map between Fock spaces with different eigenvalues of $A[0]$ and $B[0]$. We refer to [3] for the explicit formula. When there is a fan Σ present, one can use it to modify the definition of the vertex operators to make their results zero unless both n and $B[0]$-eigenvalue of the argument lie in the same cone. This introduces a structure of the vertex algebra (see [15]) on $\mathrm{Fock}_{M\oplus N}$ and some subspaces of it. Even though the underlying vector spaces stays the same, they are denoted by $\mathrm{Fock}^{\Sigma}_{\dots}$ to show that vertex operators act differently.

A choice of a particular hypersurface X in a family means a choice of coefficients f_m for each $m\in(\Delta,1)$. We will be concerned with a generic such choice. Also, we choose generic set of numbers g_n, one for each $n\in(\Delta^*,1)$. When the ambient variety is smooth, g plays only a token role. In the presence of singularities the role of g is more essential, they code for some strange structure that helps define the chiral de Rham complex. In [3] there are introduced operators BRST_f and BRST_g

$$\mathrm{BRST}_f=\left(\sum_{m\in(\Delta,1)}(m\cdot\Phi)\mathrm{e}^{\int m\cdot B}\right)[0]$$

$$\mathrm{BRST}_g=\left(\sum_{n\in(\Delta^*,1)}(n\cdot\Psi)\mathrm{e}^{\int n\cdot A}\right)[0]$$

which act on $\mathrm{Fock}_{M\oplus N}$, $\mathrm{Fock}^{\Sigma}_{M\oplus N}$, $\mathrm{Fock}_{M\oplus K^*}$ and $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$. Moreover, for any cone $C^*\in\Sigma$ these operators act on $\mathrm{Fock}^{\Sigma}_{M\oplus C^*}$. Both operators act as differentials of a double complex, that is they anticommute, their squares are zero, and they change eigen-values of $(\deg^*\cdot A[0], \deg\cdot B[0])$ by $(1,0)$ and $(0,1)$ respectively. Besides, they both commute with $J[0]$ and $L[0]$ defined above, so the double grading on Fock descends to the cohomology of these operators.

The following statement defines chiral de Rham complex of a hypersurface in a Gorenstein toric Fano variety.

Definition 6.4 ([3]) Sections of chiral de Rham complex over the Zariski open affine chart of the hypersurface that corresponds to the cone C^* are defined as elements of cohomology of $\mathrm{Fock}_{M\oplus C^*}$ with respect to the operator

$$\mathrm{BRST}_{f,g} = \mathrm{BRST}_f + \mathrm{BRST}_g.$$

When the ambient toric variety is nonsingular this definition coincides with the definition of Malikov, Schechtman and Vaintrob for an arbitrary choice of nonzero g_n. This construction behaves well under the localization, and it allows us to define $\mathcal{MSV}(X)$ as a quasi-loco sheaf of vertex algebras, see [3].

One of the main results of [3] is the following.

Theorem 6.5 ([3]) If the ambient variety is smooth, cohomology $H^*(\mathcal{MSV}(X))$ is isomorphic to the cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$ with respect to the operator $\mathrm{BRST}_{f,g}$.

Unfortunately, it is unclear whether this theorem holds in the general case. However, we will show below that the (graded, super) Euler characteristics of $H^*(\mathcal{MSV}(X))$ equals the Euler characteristics of the above space.

Proof of the Proposition 6.2. General theory of quasi-loco sheaves developed in [3] implies that the cohomology of $\mathcal{MSV}(X)$ can be calculated by means of the Čech complex defined by the toric affine charts. Each entry of the Čech complex is the direct sum of global sections over affine charts that correspond to cones C^* in Σ of various dimensions. This naturally leads to considering the following double complex. It has entries labeled by eigenvalues of $\deg^*\cdot A[0]+\deg\cdot B[0]$ and the position in the Čech complex. The differentials are $\mathrm{BRST}_{f,g}$ and $d_{\hat{C}ech}$. After some insertions of (-1), these differentials anticommute.

Cohomology of $\mathcal{MSV}(X)$ is defined as the repeated cohomology of this complex when you first take cohomology with respect to $\mathrm{BRST}_{f,g}$ and then use Čech differential. On the other hand, one can start with cohomology of Čech differential and then do $\mathrm{BRST}_{f,g}$. It is easy to show that cohomology of the Čech differential is non-zero only at the zeroth column, where it is isomorphic to $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$, and the cohomology of the total complex is the cohomology of $\mathrm{Fock}_{M\oplus K^*}$ with respect to $\mathrm{BRST}_{f,g}$. Therefore, there exists a spectral sequence from $H^*(\mathcal{MSV}(X))$ to this space. If the ambient variety is smooth, then it is proved in [3] to degenerate immediately, providing an explicit description of the cohomology of $\mathcal{MSV}(X)$, see Theorem 6.5. Since the differentials of this spectral sequence change parity and commute with

$L[0]$ and $J[0]$, they will have no effect on the supertrace used to define $Ell(y,q)$. So we have

$$Ell(y,q) = y^{-d/2}\mathrm{SuperTrace}_{H^*\mathrm{Fock}^{\Sigma}_{M\oplus K^*}} y^{J[0]}q^{L[0]}$$

where H^* denotes cohomology with respect to $\mathrm{BRST}_{f,g}$. We have used the fact that each of the bigraded components of $\mathcal{MSV}(X)$ has finite-dimensional cohomology.

To calculate the supertrace above, we consider another double complex where the entries are $(\deg^* \cdot A[0], \deg \cdot B[0])$ eigenspaces of $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$ and the differentials are BRST_f and BRST_g. Eigenvalues of $\deg \cdot B[0]$ are non-negative, because $\deg \cdot K^* \geq 0$, but eigenvalues of $\deg^* \cdot A[0]$ are not. The supertrace over the cohomology of the double complex equals to the supertrace over the repeated cohomology if we are able to show that the corresponding spectral sequence converges. Notice that we can split the double complex according to eigenvalues of $L[0]$ and $J[0]$ because these operators commute with differentials. We take cohomology with respect to BRST_g first. One can show that this cohomology is zero for all sufficiently big values of $\deg \cdot B[0]$, see [3], which implies that the spectral sequence degenerates after a finite number of steps. Moreover, for a fixed pair of eigen-values of $L[0]$ and $J[0]$ all vertical cohomology spaces are finite-dimensional (we will show it below), so we can calculate the super-trace on the first layer of the spectral sequence.

Lemma 6.6 For a fixed pair of eigenvalues of $L[0]$ and $J[0]$ the corresponding eigenspace of BRST_g cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$ or $\mathrm{Fock}_{M\oplus K^*}$ is finite-dimensional.

Proof of the lemma. Consider the grading operator $L[0] + J[0]$. It counts the opposite of the sum of mode numbers plus the number of Φ minus the number of Ψ plus $\deg \cdot n$.

If we denote by $\mathbf{P}$ and L the ambient toric variety and the canonical line bundle over it, then cohomology of $\mathcal{MSV}(L)$ is precisely the BRST_g cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$. It was proved in [3] that $\mathcal{MSV}(L)$ is a loco sheaf with respect to the above grading operator $L[0]$. Therefore, its cohomology has a filtration such that all the quotients are finitely generated modules over $\mathbf{C}[K]$, where the action of $\mathbf{C}[K]$ is given by $\mathrm{e}^{\int m\cdot B}[0]$. However, a multiplication by $\mathrm{e}^{\int m\cdot B}[0]$ decreases the eigenvalue of $J[0]$ by $\deg^* \cdot m$, which shows that for a fixed value of $L[0]$ and $J[0]$ the eigen-space is finite-dimensional. A case of $\mathrm{Fock}_{M\oplus K^*}$ is treated analogously, because it admits a filtration whose quotients are finitely generated $\mathbf{C}[K]$ modules. $\square$

So now we can take supertrace over the cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$ by BRST_g. We notice that this differential commutes with $A[0]$ so we can split the whole picture according to eigenvalues of $A[0]$. Basically, we have managed to move the problem from the Calabi-Yau hypersurface to the line bundle over the ambient toric variety which has a huge torus symmetry.

Let $m \in M$ be one such eigenvalue. We claim that for fixed values of m, $J[0]$ and $L[0]$ the dimension of the corresponding eigen-space of $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$ is finite. Really, look at the $L[0] + rJ[0]$ eigenvalue for sufficiently big fixed r, which is chosen in a way that $m \cdot n + (r-1)\deg \cdot n \geq 0$ for all $n \in K^*$. For any element $|m,n>$ this eigenvalue is bounded from below, and only finitely many n work for each particular

value. Moreover, multiplying by $A, B, \Phi, \Psi[-l]$ only increases this value, except for a finite number of anticommuting modes $\Psi[-l]$, which can decrease the eigenvalue by no more than a constant.

As a result, for a fixed m we can calculate the supertrace over the BRST_g cohomology of $\mathrm{Fock}_{m\oplus K^*}$ directly over the Fock space. We get

$$\mathrm{SuperTrace}_{\mathrm{Fock}_{m\oplus K^*}} y^{J[0]}q^{L[0]} = \mathrm{SuperTrace}_{\mathrm{Fock}_{m\oplus K^*}} (yq^{-r})^{J[0]}q^{L[0]+rJ[0]} =$$

$$= \left(\sum_{n\in K^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*}\right) G(y,q)^{d+2}$$

where

$$G(y,q) = \prod_{k\geq 1} \frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}.$$

Really, for every $n \in K^*$ the space $\mathrm{Fock}_{m\oplus n}$ is isomorphic to the tensor product of spaces

$$\mathbf{C} \oplus m_i \cdot \Phi[-k]\mathbf{C}, k \geq 0$$

$$\mathbf{C} \oplus n_i \cdot \Psi[-k]\mathbf{C}, k > 0$$

$$\mathbf{C} \oplus m_i \cdot B[-k]\mathbf{C} \oplus (m_i \cdot B[-k])^2\mathbf{C} \oplus ..., \ k > 0$$

$$\mathbf{C} \oplus n_i \cdot A[-k]\mathbf{C} \oplus (n_i \cdot A[-k])^2\mathbf{C} \oplus ..., \ k > 0$$

where $i = 1..d$ and $\{m_i\}$ and $\{n_i\}$ are arbitrary bases of $M_{\mathbf{C}}$ and $N_{\mathbf{C}}$. When one calculates the supertrace, the first two spaces contribute to the numerator and the last two to the denominator of $G(y,q)$. The term

$$y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*}$$

appears when one evaluates $J[0]$ and $L[0]$ on $|m,n>$ itself.

We remark that it is legal to multiply these two double series above because the former has only a finite number of entries for each $L[0] + rJ[0]$ eigenvalue and coefficients of the second one are supported inside a parabola. Moreover, our arguments show that the resulting double series in y and q will contain entries of a fixed bi-degree only for a finite number of m.

When we add the supertraces above for all m we recover the formula of Proposition 6.2, which finishes its proof. □

Our goal now is to compare the elliptic genera for dual Calabi-Yau hypersurfaces. This means switching M, K and N, K^*. One would expect to find a direct argument based simply on the formula of Proposition 6.2, but we were unable to find one. The key result of [3] which we will use here is

Theorem 6.7 ([3]) Cohomology of $\mathrm{Fock}_{M\oplus K^*}$ with respect to $\mathrm{BRST}_{f,g}$ is isomorphic to the cohomology of $\mathrm{Fock}_{K\oplus N}$ with respect to $\mathrm{BRST}_{f,g}$.

We remark that this statement is by no means obvious. Also, $\mathrm{Fock}_{K\oplus N}$ is not exactly the analog of $\mathrm{Fock}_{M\oplus K^*}$ for the dual pair. The difference comes from the fact that in the formula (12) only negative modes of Ψ are allowed, as opposed to all non-positive modes of Φ. In the mirror picture non-positive modes of Ψ and negative modes of Φ appear. However, one can still construct an isomorphism between $\mathrm{Fock}_{K\oplus N}$ and the mirror analog of $\mathrm{Fock}_{M\oplus K^*}$ by shifting the mode numbers of Φ and Ψ by -1 and 1 respectively. Under this isomorphism, $L[0]$ and $J[0]$ are related to their mirror versions as follows

$$L_X[0] = L_{X^*}[0] + J_{X^*}[0],\ \ J_X[0] = -J_{X^*}[0]. \tag{13}$$

This is easy to check from the description of the action of $J[0]$ and $L[0]$ on the monomials in (12).

Proposition 6.8 Let X and X^* be two mirror hypersurfaces in Gorenstein toric Fano varieties. Then

$$Ell(X;y,q) = y^{-d}q^{d/2}Ell(X^*;y^{-1}q,q).$$

Proof. The key observation here is that the supertraces calculated over the BRST cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^*}$ and $\mathrm{Fock}_{M\oplus K^*}$ are the same. Really, we can prove the same supertrace formula for the $\mathrm{BRST}_{f,g}$ cohomology of $\mathrm{Fock}_{M\oplus K^*}$. On the other hand, Theorem 6.7 shows that the cohomology of $\mathrm{Fock}_{M\oplus K^*}$ is isomorphic as a vector space to the cohomology of $\mathrm{Fock}_{K\oplus N}$ for the same map. It was important here that we got rid of the fans. Now we use formula (13) to get

$$Ell(X;y,q) = y^{-d/2}\mathrm{SuperTrace}_{H^*\mathrm{Fock}_{M\oplus K^*}} y^{J_X[0]}q^{L_X[0]}$$

$$= y^{-d/2}\mathrm{SuperTrace}_{H^*\mathrm{Fock}_{K\oplus N}} y^{-J_{X^*}[0]}q^{L_{X^*}[0]+J_{X^*}[0]} = y^{-d}q^{d/2}Ell(X^*;y^{-1}q,q).$$

□

Remark 6.9 When X (or X^*) is smooth, we can use the fact that $Ell(y,q)$ is a weak Jacobi form to show that

$$Ell(X;y,q) = (-1)^d Ell(X^*;y,q)$$

as predicted by Mirror Symmetry.

7 Modular properties of elliptic genera of singular varieties

We will now show that $Ell(X;y,q)$ is a weak Jacobi form for an arbitrary reflexive polytope Δ. This will allow us to prove mirror duality of elliptic genera of hypersurfaces in full generality. In addition we will extend the definition of elliptic genera for toric varieties to the Gorenstein case and prove their transformation properties under $\Gamma_0(2)$.

Theorem 7.1 Elliptic genus of a generic Calabi-Yau hypersurface of dimension d in any toric Gorenstein Fano variety is a weak Jacobi form of weight 0 and index $d/2$ (with a character if d is odd).

Proof. We assume that the fan Σ is simplicial, which we can always do because the formula of Proposition 6.2 holds for any simplicial subdivision of faces of Δ^*. Notice that $Ell(y,q)$ as defined in that proposition is the value at $\nu = 0$ of

$$\rho(y,q,\nu) = y^{-d/2} \sum_{m \in M} \mathrm{e}^{2\pi \mathrm{i} m\cdot\nu} \left(\sum_{n\in K^*} y^{n\cdot \deg - m\cdot \deg^*} q^{m\cdot n + m\cdot \deg^*} G(y,q)^{d+2} \right)$$

where $\nu \in N_{\mathbf{C}}$. At each bidegree of $q^k y^l$ the corresponding coefficient is a linear combination of a finite number of exponential functions $\mathrm{e}^{2\pi \mathrm{i} m\cdot\nu}$.

The idea is to split this formula for ρ into the sum over the cones of Σ of maximum dimension. In the smooth case it is precisely Bott formula.

Lemma 7.2 For each cone $C^* \in \Sigma$ of maximum dimension we denote the generators of its one-dimensional faces by n_i, $i = 1, ..., d+2$ suppressing the dependence on C^*. We also denote the dual basis in $M_{\mathbf{Q}}$ by $\{m_i\}$. We choose $n_1 = \deg^*$. Elements of the group $G = N/(\mathbf{Z}n_1 + ... + \mathbf{Z}n_{d+2})$ could be identified with lattice points in C^* whose coordinates in the basis $\{n_i\}$ are less than 1. We call this set of points $\mathrm{Box}(C^*)$. Then

$$\rho(y,q,\nu) = \sum_{C^*\in\Sigma, \dim C^* = d+2} \frac{1}{|G|} \sum_{n,l\in \mathrm{Box}(C^*)} y^{n\cdot \deg} \frac{\theta(m_1\cdot\nu,\tau)}{\theta(m_1\cdot\nu - z,\tau)} \times$$
$$\times \prod_{i=2}^{d+2} \frac{\theta(-m_i\cdot\nu - m_i\cdot l - (m_i\cdot n)\tau - z)}{\theta(-m_i\cdot\nu - m_i\cdot l - (m_i\cdot n)\tau)}.$$

Proof of the lemma. First of all, we must explain what the expression above means. It is a finite sum of meromorphic functions of $(\tau, z, \nu) \in H \times \mathbf{C} \times N_{\mathbf{C}}$. Most terms of the summation are not defined at $\nu = 0$. We also notice that m_1 is orthogonal to all points of the $\mathrm{Box}(C^*)$ because Δ^* is reflexive.

We start by rewriting ρ as the sum over cones of any dimension which is analogous to the formulas of Section 5. We have

$$\rho(y,q,\nu) = y^{-\frac{d}{2}} \sum_{m\in M} \mathrm{e}^{2\pi \mathrm{i} m\cdot\nu} \sum_{C^*\in\Sigma} (-1)^{\mathrm{codim} C^*} \sum_{n\in C^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot \deg^*} G(y,q)^{d+2}$$

where the summation is taken over all cones in Σ that contain $\deg^*$. This assures that each point $n \in K^*$ contributes once. Our next goal is to somehow get rid of cones of positive codimension. For each of these cones there is an element $m_{C^*} \in M_1$ orthogonal to C^*. Therefore, for elements of m that differ by m_{C^*} the corresponding terms in the above expression for ρ differ only by $\mathrm{e}^{2\pi \mathrm{i} m_{C^*}\cdot\nu}$. As a result,

$$\rho(y,q,\nu) \prod_{C^*, \mathrm{codim} C^* > 0} (1 - \mathrm{e}^{2\pi\mathrm{i} m_{C^*}\cdot\nu}) = y^{-\frac{d}{2}} \sum_{m\in M} \mathrm{e}^{2\pi \mathrm{i} m\cdot\nu} \sum_{C^*\in\Sigma, \dim C^* = d+2}$$

$$\Big(\sum_{n\in C^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n+m\cdot\deg^*} G(y,q)^{d+2}\Big) \prod_{C_1^*,\mathrm{codim} C_1^*>0} (1-\mathrm{e}^{2\pi\mathrm{i} m_{C_1^*}\cdot\nu}).$$

This identity should be understood as an identity in $\mathbf{C}[M][y,y^{-1}][[q]]$. However, we can interpret both sides as double series in y and q whose coefficients are meromorphic functions on $N_{\mathbf{C}}$. Really, for each C^* its contribution to ρ is the supertrace over $H^0(\pi_*\mathcal{MSV}(L),A_{\mathbf{C}})$ of $y^{J[0]}q^{L[0]}\mathrm{e}^{2\pi\mathrm{i}A[0]}$. It was shown in [3] that for fixed powers of y and q the sections of $\pi_*\mathcal{MSV}(L)$ form a Noetherian almost-module over $\mathbf{C}[C\cap M_1]$. As a result, the coefficients by y^aq^b in the above supertrace are Hilbert functions of some finitely generated modules over $\mathbf{C}[C\cap M_1]$ and are therefore finite linear combinations of $\mathrm{e}^{2\pi\mathrm{i}m\cdot\nu}$ over products of $(1-\mathrm{e}^{2\pi\mathrm{i}k_im_i\cdot\nu})$ where k_im_i are generators of one-dimensional faces of $C\cap M_1$. We remark, that this happens only after you multiply by $G(y,q)^{d+2}$, otherwise many other m seem to contribute to a given coefficient by y^aq^b.

Now it remains to show that for each cone C^* we have

$$\sum_{m\in M}\sum_{n\in C^*} \mathrm{e}^{2\pi\mathrm{i}m\cdot\nu} y^{n\cdot\deg-m\cdot\deg^*} q^{m\cdot n+m\cdot\deg^*} G(y,q)^{d+2} =$$

$$=\frac{1}{|G|}\sum_{n,l\in\mathrm{Box}(C^*)} y^{n\cdot\deg}\frac{\theta(m_1\cdot\nu,\tau)}{\theta(m_1\cdot\nu-z,\tau)}\prod_{i=2}^{d+2}\frac{\theta(-m_i\cdot\nu-m_i\cdot l-(m_i\cdot n)\tau-z)}{\theta(-m_i\cdot\nu-m_i\cdot l-(m_i\cdot n)\tau)}.$$

There are two ways to do so. One can use the isomorphism between BRST_g cohomology of $\mathrm{Fock}_{M\oplus C^*}$ and the direct sum over all n in $\mathrm{Box}(C^*)$ of the spaces G-invariant sections of the flat space Fock space found in [3]. Alternatively, a more elementary argument is to write

$$\sum_{m\in M}\mathrm{e}^{2\pi\mathrm{i}m\cdot\nu}\ldots=\frac{1}{|G|}\sum_{l\in\mathrm{Box}(C^*)}\sum_{m\in\mathbf{Z}\{m_i\}}\mathrm{e}^{2\pi\mathrm{i}m\cdot\nu+m\cdot l}\ldots$$

and then notice that

$$\sum_{n\in C^*}\ldots=\sum_{n\in\mathrm{Box}(C^*)}\sum_{n'\in\mathbf{Z}_{\geq}0\{n_i\}}\ldots$$

Afterwards, the key formula (11) allows us to rewrite ρ as the sum of infinite products which are easily seen to coincide with ratios of theta functions. This finishes the proof of the lemma. $\square$

We can now go back to the proof of Proposition 7.1. It is enough to check modular properties of $Ell(y,q)$ for generators of the Jacobi group. This follows from the transformation properties of the theta function and Lemma 7.2. We will sketch the argument in the hardest case of $(z,\tau)\to(\frac{z}{\tau},-\frac{1}{\tau})$. Consider the change in ρ incurred when we change $z\to\frac{z}{\tau},\nu\to\frac{\nu}{\tau},\tau\to-\frac{1}{\tau}$. We have

$$\rho(\frac{z}{\tau},-\frac{1}{\tau},\frac{\nu}{\tau})=\sum_{C^*\in\Sigma,\dim C^*=d+2}\frac{1}{|G|}\sum_{n,l\in\mathrm{Box}(C^*)}\mathrm{e}^{\frac{2\pi\mathrm{i}zn\cdot\deg}{\tau}}\frac{\mathrm{e}^{\pi\mathrm{i}\frac{(m_1\cdot\nu)^2}{\tau}}}{\mathrm{e}^{\pi\mathrm{i}\frac{(m_1\cdot\nu-z)^2}{\tau}}}\frac{\theta(m_1\cdot\nu,\tau)}{\theta(m_1\cdot\nu-z,\tau)}\times$$

$$\times\prod_{i=2}^{d+2}\frac{\theta(-\frac{m_i\cdot\nu}{\tau}-m_i\cdot l+\frac{m_i\cdot n}{\tau}-\frac{z}{\tau},-\frac{1}{\tau})}{\theta(-\frac{m_i\cdot\nu}{\tau}-m_i\cdot l+\frac{m_i\cdot n}{\tau},-\frac{1}{\tau})}=$$

$$= \sum_{C^*\in\Sigma,\dim C^*=d+2} \frac{1}{|G|} \sum_{n,l\in\mathrm{Box}(C^*)} \mathrm{e}^{\frac{2\pi\mathrm{i} nz\cdot\deg}{\tau}} \mathrm{e}^{\pi\mathrm{i}\frac{(m_1\cdot\nu)^2}{\tau}-\pi\mathrm{i}\frac{(m_1\cdot\nu-z)^2}{\tau}} \frac{\theta(m_1\cdot\nu,\tau)}{\theta(m_1\cdot\nu-z,\tau)}\times$$

$$\times\prod_{i=2}^{d+2} \frac{\theta(-m_i\cdot\nu-m_i\cdot l\tau+m_i\cdot n-z,\tau)}{\theta(-m_i\cdot\nu-m_i\cdot l\tau+m_i\cdot n,\tau)} \mathrm{e}^{\frac{\pi\mathrm{i}}{\tau}((m_i\cdot\nu+m_i\cdot l\tau+z-m_i\cdot n)^2-(m_i\cdot\nu+m_i\cdot l\tau-m_i\cdot n)^2)} =$$

$$= \mathrm{e}^{\frac{d\pi\mathrm{i}z^2}{\tau}} \mathrm{e}^{\frac{2\pi\mathrm{i}z}{\tau}(\deg\cdot\nu)} \sum_{C^*} \frac{1}{|G|} \sum_{n,l} \mathrm{e}^{2\pi\mathrm{i}l\cdot\deg z} \frac{\theta(m_1\cdot\nu,\tau)}{\theta(m_1\cdot\nu-z,\tau)}\times$$

$$\times\prod_{i=2}^{d+2} \frac{\theta(-m_i\cdot\nu-m_i\cdot l\tau+m_i\cdot n-z,\tau)}{\theta(-m_i\cdot\nu-m_i\cdot l\tau+m_i\cdot n,\tau)}.$$

In the last step we have used that m_1 is orthogonal to all elements of Box and $\sum_1^{d+2} m_i = \deg$. Notice that l and n are switched now. Also, the change of sign of n is not important, because n could be really thought of as the element of $N/\mathbf{Z}\{n_i\}$. This implies that

$$\rho(\frac{z}{\tau},-\frac{1}{\tau},\frac{\nu}{\tau}) = \mathrm{e}^{\frac{d\pi\mathrm{i}z^2}{\tau}} \mathrm{e}^{\frac{2\pi\mathrm{i}z}{\tau}(\deg\cdot\nu)}\rho(z,\tau,\nu)$$

which gives the desired modular property for $Ell(y,q)$ once we plug in $\nu=0$. $\square$

We can combine the results of Theorem 7.1 and Proposition 6.8 to prove mirror duality of elliptic genera of Calabi-Yau hypersurfaces in arbitrary Gorenstein toric Fano varieties.

Theorem 7.3 Let X and X^* be two mirror hypersurfaces in Gorenstein toric Fano varieties. Then

$$Ell(X;y,q) = (-1)^d Ell(X^*;y,q).$$

Proof. Use the result of 6.8 and transformation property of Ell under $(y,q)\to(y^{-1}q,q)$. $\square$

We will now extend the results of Section 5 from smooth toric varieties to toric varieties with Gorenstein singularities. We will return to the setup of that section, that is we have two dual lattices of M and N of rank d and a complete fan Σ in N which defines the toric variety $\mathbf{P}$. We first notice that (in the smooth case) the formula of Theorem 5.5 could be rewritten as

$$Ell(\mathbf{P},y,q) = y^{-d/2}\sum_{m\in M}\sum_{C^*\in\Sigma}(-1)^{\mathrm{codim}C^*}(\sum_{n\in C^*} q^{m\cdot n}y^{\deg\cdot n})G(y,q)^d$$

where $\deg\cdot n$ is a piecewise linear function on N which equals 1 on the generators of one-dimensional faces of Σ. In general $\mathbf{P}$ has only Gorenstein singularities if and only if this function takes integer values. This prompts the following definition.

Definition 7.4 For a toric Gorenstein variety $\mathbf{P}$ we define

$$Ell(\mathbf{P},y,q) = y^{-d/2}\sum_{m\in M}\sum_{C^*\in\Sigma}(-1)^{\mathrm{codim}C^*}(\sum_{n\in C^*} q^{m\cdot n}y^{\deg\cdot n})G(y,q)^d.$$

We remark that the expression above should be interpreted as follows. While the series $\sum_{n\in C^*} q^{m\cdot n} y^{\deg\cdot n}$ in general diverges at $q=0$, if we consider the product of this double series with $G(y,q)$ one can show that the result will only have non-negative powers of q. Alternatively, one can notice that for any given m and C^* the function $\sum_{n\in C^*} q^{m\cdot n} y^{\deg\cdot n}$ is a rational function of q,y which could be expanded around $q=0$. Besides, it could be shown that Definition 7.4 coincides with Definition 6.1 when one defines $\mathcal{MSV}(\mathbf{P})$ as in [3]. The proof is very similar to the hypersurface case.

Since a compact toric variety is never Calabi-Yau, we do not expect $Ell(\mathbf{P},y,q)$ to be a Jacobi form. However we will now show that $(-1)^{d/2} Ell(\mathbf{P},-1,q)$ has expected modular properties with respect to $\Gamma_0(2)$. The idea is the same as in the hypersurface case. We consider

$$\rho(q,\nu) = \sum_{m\in M} \mathrm{e}^{m\cdot\nu} \sum_{C^*\in\Sigma} (-1)^{\mathrm{codim} C^*} (\sum_{n\in C^*} q^{m\cdot n} (-1)^{\deg\cdot n}) G(-1,q)^d \tag{14}$$

as a function on $H\times N_{\mathbf{C}}$. We assume that Σ is simplicial, which could be done safely, because one can show that ρ does not change under crepant subdivisions of Σ. Then we rewrite it as

$$\rho(q,\nu) = \sum_{C^*\in\Sigma, \mathrm{dim} C^*=d} \frac{1}{|G|} \sum_{k,l\in \mathrm{Box}(C^*)} (-1)^{\deg\cdot k} \prod_{i=1}^{d} \frac{\theta(\frac{1}{2} - m_i\cdot k\tau - m_i\cdot\nu - m_i\cdot l,\tau)}{\theta(-m_i\cdot k\tau - m_i\cdot\nu - m_i\cdot l,\tau)}. \tag{15}$$

Here m_i denote the basis of $M_{\mathbf{C}}$ dual to the basis of generators n_i of one-dimensional faces of C^*. In general $m_i\notin M$. The proof of this formula is completely analogous to the proof of Lemma 7.2 so we skip it.

It is well-known that the group $\Gamma_0(2)$ is generated by $\tau\to\tau+1$ and $\tau\to \frac{\tau}{-2\tau+1}$. Clearly, $Ell(\mathbf{P},-1,\tau)$ is not affected by the first transformation. We will now calculate how it changes under the second one.

Proposition 7.5

$$Ell(\mathbf{P},-1,\frac{\tau}{-2\tau+1}) = (-i)^d Ell(\mathbf{P},-1,\tau)$$

Proof. It is easy to derive that

$$\theta(\frac{z}{-2\tau+1},\frac{\tau}{-2\tau+1}) = -\mathrm{i}\sqrt{2\tau-1}\,\mathrm{e}^{\frac{2\pi\mathrm{i}z^2}{2\tau-1}}\,\theta(z,\tau).$$

It implies

$$\rho(\frac{\tau}{-2\tau+1},\frac{\nu}{-2\tau+1}) = \sum_{C^*\in\Sigma, \mathrm{dim} C^*=d} \frac{1}{|G|} \sum_{k,l\in \mathrm{Box}(C^*)} (-1)^{\deg\cdot k}$$

$$\prod_{i=1}^{d} \frac{\theta(\frac{1}{2} - m_i\cdot k\frac{\tau}{1-2\tau} - m_i\cdot\frac{\nu}{1-2\tau} - m_i\cdot l,\frac{\tau}{1-2\tau})}{\theta(-m_i\cdot k\frac{\tau}{1-2\tau} - m_i\cdot\frac{\nu}{1-2\tau} - m_i\cdot l,\frac{\tau}{1-2\tau})} =$$

$$\sum_{C^*\in\Sigma,\dim C^*=d}\frac{1}{|G|}\sum_{k,l\in\mathrm{Box}(C^*)}(-1)^{\deg\cdot k}\prod_{i=1}^d\frac{\theta(\frac{-\tau+\frac12-m_i\cdot k\tau-m_i\cdot\nu-m_i\cdot l(1-2\tau)}{1-2\tau},\frac{\tau}{1-2\tau})}{\theta(\frac{-m_i\cdot k\tau-m_i\cdot\nu-m_i\cdot l(1-2\tau)}{1-2\tau},\frac{\tau}{1-2\tau})}=$$

$$\sum_{C^*,\dim C^*=d}\frac{1}{|G|}\sum_{k,l\in\mathrm{Box}(C^*)}(-1)^{\deg\cdot k}\prod_{i=1}^d\frac{\theta(-\tau+\frac12-m_i\cdot k\tau-m_i\cdot\nu-m_i\cdot l(1-2\tau),\tau)}{\theta(-m_i\cdot k\tau-m_i\cdot\nu-m_i\cdot l(1-2\tau),\tau)}$$

$$\times\prod_{i=1}^d\mathrm{e}^{\frac{2\pi\mathrm{i}}{2\tau-1}(-\tau+\frac12)(-\tau+\frac12-2m_i\cdot k\tau-2m_i\cdot\nu-2m_i\cdot l(1-2\tau))}=$$

$$\sum_{C^*\in\Sigma,\dim C^*=d}\frac{1}{|G|}\sum_{k,l\in\mathrm{Box}(C^*)}(-1)^{\deg\cdot k}(-\mathrm{i})^d\prod_{i=1}^d\frac{\theta(\frac12-m_i\cdot(k-2l)\tau-m_i\cdot\nu-m_i\cdot l,\tau)}{\theta(-m_i\cdot(k-2l)\tau-m_i\cdot\nu-m_i\cdot l,\tau)}$$

$$=(-\mathrm{i})^d\rho(\tau,\nu).$$

At the last step we used the fact that we can consider k,l to be representatives of the group G. Really, when we change k by n which is an integer combination of n_i, the ratio of θ functions changes by $(-1)^{m_i\cdot n}$, so overall the change is $(-1)^{\deg\cdot n}$ which is compensated by the change in the factor $(-1)^{\deg\cdot k}$.

Now it remains to plug in $\nu=0$ and to use (14). $\square$

Remark 7.6 One can easily show that $Ell(\mathbf{P},-1,q)$ equals zero if the dimension of $\mathbf{P}$ is odd. Really, if we switch $k\to -k$, $l\to -l$ in the above summation and then use $\theta(\frac12-z,\tau)=-\theta(-\frac12+z,\tau)=\theta(\frac12+z,\tau)$ we will see that $\rho(-\nu,\tau)=(-1)^d\rho(\nu,\tau)$. So for odd d $Ell(\mathbf{P},-1,q)=0$. When $\mathbf{P}$ is smooth this could be also shown easily by means of Chern classes.

Theorem 7.7 Let $\mathbf{P}$ be a Gorenstein toric variety of even dimension d. Analogously to Section 5 we introduce

$$\widehat{Ell}(\mathbf{P},q)=(-1)^{d/2}Ell(\mathbf{P},-1,q)G(-1,q)^{-d}.$$

Then this normalized genus has transformation properties of the modular form of weight d with respect to the group $\Gamma_0(2)$.

Proof. We notice that

$$G(-1,q)=\eta(2\tau)^2/\eta(\tau)^4$$

where $\eta(\tau)$ is the Dedekind η-function. Then the modular transformation properties of $\widehat{ELL}$ follow from Proposition 7.5 and transformation properties of η (cf. for example [5]). $\square$

Remark 7.8 We conjecture that $\widehat{Ell}(\mathbf{P},q)$ is a modular form. In view of the above theorem, it simply means that it is holomorphic for all τ and has appropriate Fourier expansions around the cusps of $\Gamma_0(2)$.

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